Legendre Polynomials

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February 9, 1993

The Legendre polynomials are are orthogonal over $[-1, 1]$ and all their zeros lie in this interval.

The basis for Gaussian quadrature is Legendre polynomials.

In Anthony Ralston's book "A first course in Numerical Analysis" (1965) he derives the Gaussian quadrature rules starting with the Hermite interpolation formula, a formula for interpolating a function based on the value of the function and its first derivative at arbitrary points. The idea is to integrate Hermite's formula and choose the interpolation points so that the terms that depend on the derivative vanish. The fact that the nth Legendre polynomial is orthogonal to all polynomials of order less than n is a key step in this derivation.

The integration points for Gaussian quadrature are the zeros of the Legendre polynomials. The weights are determined as follows. Let a_i, $i = 1 \ldots n$ be the zeros of P_n, the nth Legendre polynomial. Then the weight w_i associated with a_i is given by

$$a_i = \int_{-1}^{1} \frac{P_n(x)}{(x - a_i) P_n'(a_i)} \, dx.$$

Note that the integrand is

$$\frac{\prod_{j \neq i}(x - a_j)}{\prod_{j \neq i}(a_i - a_j)},$$

which is the Lagrange interpolating polynomial for the points $\{a_j\}_{j \neq i}$.

The error in the Gaussian quadrature rule is

$$\frac{f^{(2n)}(\xi)}{(2n)!} \int_{-1}^{1} p_n^2(x) \, dx$$

for some $\xi \in (-1, 1)$. In particular, if f is a polynomial of degree $2n - 1$ or less, the error is zero. Here

$$p_n(x) = \prod_{i=1}^{n} (x - a_i) = \frac{2^n (n!)^2}{(2n)!} P_n(x).$$

Since

$$\int_{-1}^{1} P_m(x)P_n(x)\,dx = \frac{2}{2n+1}\delta_{mn},$$

one may derive an expression for the error strictly in terms of $f^{(2n)}(\xi)$ and n.

One may generalize this construction to weighted integrals using polynomials that are orthonogal with respect to that weight. For example, in order to evaluate integrals of the form

$$\int_0^\infty e^{-x} f(x)\,dx$$

where f is a polynomial, we develop and integration formula using Laguerre polynomials because they satisfy the orthogonality condition

$$\int_0^\infty e^{-x} L_m(x)L_n(x)\,dx = \delta_{mn}.$$

To evaluate integrals of the form

$$\int_{-\infty}^\infty e^{-x^2} f(x)\,dx$$

we use Hermite polynomials because they satisfy

$$\int_{-\infty}^\infty e^{-x^2} H_m(x)H_n(x)\,dx = \sqrt{\pi}2^n n!\delta_{mn}.$$

Suppose the polynomials $p_m(x)$ are orthogonal over (a, b) with respect to the weight $w(x) > 0$. Then all the zeros of the $p_m(x)$ are real, have multiplicity 1, and lie in (a, b).

To prove this, let $a_1 \ldots a_n$ be the places where p_m changes signs in (a, b). Then $(x - a_1)(x - a_2)\cdots(x - a_n)p_m(x)$ never changes signs in (a, b). Since p_m is orthogonal to any polynomial of degree less than m,

$$\int_a^b (x - a_1)(x - a_2)\cdots(x - a_n)p_m(x)w(x)\,dx = 0$$

unless $m = n$. But the integrand is not zero and never changes signs. Therefore it must be the case that $m = n$. Since p_m changes signs m times we know that all the roots of p_m are isolated.