Upper bounds on non-central chi-squared tails and truncated normal moments

John D. Cook
Department of Biostatistics
P. O. Box 301402, Unit 1409
The University of Texas, M. D. Anderson Cancer Center
Houston, Texas 77230-1402, USA
cook@mdanderson.org

October 2, 2010

Abstract

We show that moments of the truncated normal distribution provide upper bounds on the tails of the non-central chi-squared distribution, then develop upper bounds for the former.

1 Preliminaries

We are interested in obtaining upper bounds on $P(Y > y)$ where Y is a non-central chi-squared random variable with k degrees of freedom and non-centrality parameter λ. The density function for such a random varialble is given by

$$f(x; k, \lambda) = \frac{1}{2} \exp\left(-\frac{x+\lambda}{2}\right) \left(\frac{x}{\lambda}\right)^{\frac{k}{4}-\frac{1}{2}} I_{\frac{k}{2}-1}(\sqrt{\lambda x}) \tag{1}$$

where I_ν is a modified Bessel function of the first kind.

Ifantis and Siafarikas give the following bounds in their paper [1].

For $y > 0$ and $\nu > -1/2$,

$$1 < \Gamma(\nu+1) \left(\frac{2}{y}\right)^{\nu} I_\nu(y) < \cosh y \tag{2}$$

and for $y > x > 0$ and $\nu > -1/2$,

$$\exp(x-y)\left(\frac{x}{y}\right)^{\nu} < \frac{I_\nu(x)}{I_\nu(y)} < \left(\frac{x}{y}\right)^{\nu} \tag{3}$$

If we assume $y > x = 1$, equation (3) tells us

$$\frac{I_\nu(1)}{e} y^\nu \exp(y) > I_\nu(y) > I_\nu(1) y^\nu. \tag{4}$$

We could apply equation (2) to bound $I_\nu(1)$ if desired.

2 Upper bounds on non-central chi-squared tails

If we integrate equation (1) and apply inequality (4) we have

$$\int_y^\infty f(x; k, \lambda)\, dx \leq \frac{I_{\frac{k}{2}-1}(1)}{2\exp\left(\frac{\lambda}{2}+1\right)} \int_y^\infty \exp\left(-\frac{x}{2} + \sqrt{\lambda x}\right) x^{\frac{k}{2}-1}\, dx \tag{5}$$

provided $y > 1$ and $k > 1$.

The change of variable $x = u^2$ turns the integral above into

$$\frac{I_{\frac{k}{2}-1}(1)}{e} \int_{\sqrt{y}}^\infty \exp\left(-\frac{(u-\sqrt{\lambda})^2}{2}\right) u^{k-1}\, du.$$

This shows that the non-central chi-squared tail probability

$$\int_y^\infty f(x; k, \lambda)\, dx$$

is bounded by

$$\frac{\sqrt{\pi}}{2e}\, \Phi(\sqrt{y})\, I_{\frac{k}{2}}(1)\, M_{k-1} \tag{6}$$

where $M_{k-1} = E(X^{k-1})$ and X is a normal(μ, 1) random variable truncated to the interval $(\sqrt{y}, \infty)$.

3 Integral bounds

We now turn to finding upper bounds of the integral

$$g(x,\mu,r) = \int_x^\infty t^r \, \exp\left(-\frac{(t-\mu)^2}{2}\right) \, dt \tag{7}$$

When $x = \mu$ and r is integer-valued, the function $g(\mu, \mu, r)$ is related to several special functions including the repeated integral of the error function and parabolic cylinder functions. See [2] equations 7.2.3 and 19.14.2. However, no one has given a name to the general function $g(x, \mu, r)$. Perhaps it could be called an incomplete repeated integral of the error function, though that is a mouthful.

William A. Huber suggested using the inequality

$$t^r < x^r \exp\left(\frac{r}{x}t - r\right)$$

for sufficiently large x in order to get an upper bound on the integral in equation (7). Huber's inequality holds because the exponential of any linear function of t eventually bounds any power of t provided the leading coefficient of the linear function is positive.

This shows that for sufficiently large x,

$$g(x,\mu,r) < x^r \int_x^\infty \exp\left(-\frac{(t-\mu)^2}{2} + \frac{r}{x}t - r\right) \, dt.$$

The integrand above is a quadratic function of t and so the integral can be computed in terms of the complementary error function as

$$\sqrt{\frac{\pi}{2}} \exp\left(-r + \frac{r\mu}{x} + \frac{r^2}{2x^2}\right) \operatorname{erfc}\left(\frac{x - \mu - \frac{r}{x}}{\sqrt{2}}\right). \tag{8}$$

We can further bound equation (8) by using the following bound from [2] equation 7.1.13.

$$\operatorname{erfc}(z) \leq \frac{\sqrt{\pi}}{2} \frac{\exp(-z^2)}{z + \sqrt{z^2 + \frac{4}{\pi}}}$$

This yields

$$g(x,\mu,r) \leq \frac{\pi}{2\sqrt{2}} \exp\left(-x^2 - r + \frac{r\mu}{x} + \frac{r^2}{2x^2}\right) \frac{x^r}{x + \sqrt{x^2 + \frac{4}{\pi}}}. \tag{9}$$

4 Conclusions

The rth moment of a normal(μ, 1) distribution truncated to the interval (x, ∞) is $g(x, \mu, r)/\Phi(x)$ and so an upper bound follows directly from equation (9).

Equation (6) bounds the tails of the non-central chi-squared distribution in terms of moments of a truncated normal. Combining estimates we have

$$\begin{aligned}
\int_y^\infty f(x; k, \lambda)\, dx &\leq \frac{\sqrt{\pi}}{2e}\, \Phi(\sqrt{y})\, I_{\frac{k}{2}}(1)\, M_{k-1} \\
&= \frac{\sqrt{\pi}}{2e}\, I_{\frac{k}{2}}(1)\, g(\sqrt{y}, \sqrt{\lambda}, k-1) \\
&\leq \frac{\sqrt{\pi}}{2\sqrt{e}}\, I_{\frac{k}{2}}(1)\, \exp\left(-y - r + r\sqrt{\frac{\lambda}{y} + \frac{r^2}{2y}}\right) \frac{y^{r/2}}{\sqrt{y} + \sqrt{y + \frac{4}{\pi}}}
\end{aligned}$$

for sufficiently large y where $r = k - 1$.

References

[1] E. K. Ifantis and P. D. Siafarikas. Bounds for the modified Bessel function. *Dendiconti del Circolo Matematico di Palermo*, Serie II, Tomo XL (1991), pp 347-356.

[2] Milton Abramowitz and Irene Stegun. *Handbook of Mathematical Functions*, Dover (1972)