Chebyshev Polynomials

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Abstract

The Chebyshev polynomials are both elegant and useful. This note summarizes some of their elementary properties with brief proofs.

1 Cosines

We begin with the following identity for cosines.

$$\cos((n+1)\theta) = 2\cos(\theta)\cos(n\theta) - \cos((n-1)\theta) \tag{1}$$

This may be proven by applying the identity

$$\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$$

with $\alpha = n\theta$ and with $\beta = \theta$ and $\beta = -\theta$, adding equations, and rearranging terms.

Next, we claim that for each non-negative integer n, there exist integers c_i such that

$$\cos n\theta = \sum_{i=0}^{n} c_i \cos^i(\theta) \tag{2}$$

The claim is clearly true for $n = 0$ or $n = 1$. We use induction and equation (1) to establish the claim in general.

2 Chebyshev polynomials

2.1 Definition

Equation (2) says that $\cos(n\theta)$ is a polynomial in $\cos\theta$. For fixed n, we define the nth Chebyschev polynomial to be this polynomial, *i.e.*

$$\cos(n\theta) = T_n(\cos\theta). \tag{3}$$

By letting $x = \cos\theta$, this shows

$$T_n(x) = \cos(n\arccos(x)) \tag{4}$$

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for x in $[-1, 1]$.

One interpretation of equation (4) is the following quote from Forman S. Acton's book Numerical Methods that Work:

> [Chebyschev polynomials] are actually cosine curves with a somewhat disturbed horizontal scale, but the vertical scale has not been touched.

2.2 Maximum values

Several properties are immediate from equation (4). For one,

$$\max_{-1 \le x \le 1} T_n(x) = 1. \tag{5}$$

Clearly the maximum is no more than 1 since for $-1 \le x \le 1$ $T(x)$ is defined as the cosine of an argument. In fact equality holds since the maximum is attained at $x = \cos(k\pi/n)$, for $k = 1 \ldots n$.

2.3 Composition

We have the following formula for composing Chebychev polynomials

$$T_m(T_n(x)) = T_{mn}(x) \tag{6}$$

since

$$\cos(m \arccos(\cos(n \arccos(x)))) = \cos(mn \arccos(x)).$$

2.4 Zeros

From equation (4) we can determine that for $k = 1, 2, \ldots, n$, the x_n defined by

$$x_k = \cos\left(\frac{(2k-1)\pi}{2n}\right). \tag{7}$$

are zeros of T_n. Since T_n is an n^{th} degree polynomial these must be all the zeros. In particular, all the roots of T_n are real and lie in the interval $[-1, 1]$.

2.5 Recurrence relation

The recurrence relation for cosines, equation (1), leads directly to the Chebyschev recurrence relation

$$T_{n+1}(x) = 2xT_n(x) - T_{n-1}(x). \tag{8}$$

It is clear that $T_0(x) = 1$ and $T_1(x) = x$. The other T_n's can be found from equation (8). It also follows from equation (8) that for $n \ge 1$

$$T_n(x) = 2^{n-1}x^n + \mathcal{O}(x^{n-1}). \tag{9}$$

3 Differential Equation

One can show that

$$T_n(-x) = (-1)^n T_n(x) \tag{10}$$

or in other words, the even degree Chebyschev polynomials are even functions and the odd degree Chebyschev polynomials are odd functions. This follows immediately from equation (8) and induction.

One can show that T_n satisfies the following differential equation for $n \geq 1$.

$$(1 - x^2){T_n}''(x) - x{T_n}'(x) + n^2 T_n(x) = 0 \tag{11}$$

If we solve equation (11) by the power series method, we assume a solution of the form $y = \sum_{k=0}^{n} t_k x^k$ and find that the coefficients t_k must satisfy the recurrence relation

$$(n^2 - k^2)t_k + (k+1)(k+2)t_{k+2} = 0 \tag{12}$$

Since we know $t_n = 2^{n-1}$, we may work our way backward to find the other t_k's. We arrive at the formula

$$t_{n-2m} = (-1)^m 2^{n-2m-1} \frac{n}{n-m} \binom{n-m}{m} \tag{13}$$

for $m = 0, 1, \ldots \lfloor n/2 \rfloor$.

4 Extremal Properties

A monic polynomial is a polynomial whose leading coefficient is 1. In approximation theory, it is useful to identify the n^{th} degree monic polynomial with the smallest uniform norm on $[-1, 1]$, which turns out to be $2^{1-n}T_n$.

To prove this statement, let $T(x) = 2^{1-n}T_n$ and let $P_n(x)$ be an nth degree monic polynomial. Assume $|P_n(x)| < 1$ on $[-1, 1]$. Let $P_{n-1} = P_n(x) - T(x)$. Since the x^n terms cancel out, P_{n-1} is a polynomial of degree no more than $n-1$. Since T alternates $n+1$ times between the values 1 and -1, P_{n-1} changes must have at least n zeros, an impossibility for an $n-1$ degree polynomial.

5 Orthogonality

The integral

$$\int_0^\pi \cos(m\theta)\cos(n\theta)\, d\theta$$

are zero unless $m = n$. If $m = n = 0$ the integral is π, else the integral is $\pi/2$.

The change of variables $x = \cos\theta$ shows

$$\int_0^\pi \cos(m\theta)\cos(n\theta)\, d\theta = \int_{-1}^{1} T_n(x)T_m(x) \frac{dx}{\sqrt{1-x^2}}$$

and thus the Chebyschev polynomials are orthogonal over $[-1, 1]$ with respect to the weight $(1-x^2)^{-1/2}$. Further, the sequence $\frac{1}{\pi}T_0$, $\frac{2}{\pi}T_1$, $\frac{2}{\pi}T_2$, $\frac{2}{\pi}T_3 \ldots$ is an orthonormal system.

The Chebyschev polynomials also satisfy a discrete orthogonality condition, which, not surprisingly, follows directly from the analogous condition for cosines. Let x_j be the roots of T_N. Then the sum

$$\sum_{k=1}^{N} T(mx_k)\,T(nx_k)$$

is zero if $m \neq n$, N if $n = m = 0$, and $N/2$ otherwise.

6 Generating Function

The generating function for Chebyschev polynomials is given as follows

$$\frac{1-tx}{1-2tx+t^2} = \sum_{n=0}^{\infty} T_n(x)t^n. \tag{14}$$

The proof consists of letting $x = \cos\theta$ and taking the real part of both sides of the geometric series

$$\frac{1}{1-te^{i\theta}} = \sum_{n=0}^{\infty} (te^{i\theta})^n.$$