Distributed Systems of PDE in Hilbert Space

John D. Cook and R. E. Showalter

Department of Mathematics
The University of Texas at Austin
Austin, TX 78712-1082, U.S.A.

1. Introduction.

Various models of mass transport and diffusion through heterogeneous media lead to systems of partial differential equations which share a rather general structure. Among the most successful of these are the *dual porosity* models, and these vary considerably in complexity. The simplest of these are the *parallel flow* models consisting of two independent flow equations coupled by an exchange proportional to difference in pressures in the two components. These include the parabolic system

$$\frac{\partial}{\partial t}(u_1) - \vec{\nabla} \cdot A(\vec{\nabla} u_1) + a_0(u_1) + \frac{1}{\varepsilon}(u_1 - u_2) = f_1 \tag{1.1.a}$$

$$\frac{\partial}{\partial t}(u_2) - \vec{\nabla} \cdot B(\vec{\nabla} u_2) + b_0(u_1) + \frac{1}{\varepsilon}(u_2 - u_1) = f_2 \ . \tag{1.1.b}$$

and the *first-order kinetic* models, e.g., the above with $B = 0$. In order to include the geometric effects of an intricate interface between the components, one uses *distributed microstructure models*. Here a single macroscopic flow equation is coupled to a continuum of flow equations, one at each point in space representing adsorption and internal flow or reaction in a corresponding adsorption site. An example is the following system. The macroscopic flow is given by

$$\frac{\partial}{\partial t}\big(u(x,t)\big) - \vec{\nabla} \cdot A(x, \vec{\nabla} u) + q(x,t) = f(x,t) \ , \qquad x \in \Omega \ , \tag{1.2.a}$$

where $q(x,t)$ is the exchange term representing the flow into the cell Ω_x located at x. The flow within the local cell Ω_x is described in the micro-scale variable y by

$$\frac{\partial}{\partial t}\big(U(x,y,t)\big) - \vec{\nabla}_y \cdot B(x,y,\vec{\nabla}_y U) = F(x,y,t) \ , \qquad y \in \Omega_x \ . \tag{1.2.b}$$

Because of the smallness of the cells, the global pressure is assumed to be well approximated by the "constant" value $u(x,t)$ at every point of the cell boundary, so the effect of the fissures on the cell pressure is given by the interface condition

$$B(x,s,\vec{\nabla}_y U) \cdot \vec{\nu}_x + \frac{1}{\varepsilon}(U - u) = 0 \ , \qquad s \in \Gamma_x \ , \tag{1.2.c}$$

where $\vec{\nu}_x$ is the unit outward normal on the cell boundary Γ_x. We shall show below that when $\varepsilon \to 0$, this converges to the "matched" boundary condition, $u(x,t) = U(x,s,t)$ on Γ_x. Finally, the amount of fluid flux across the interface scaled by the cell size determines the remaining term in (1.2.a) by

$$q(x,t) = \frac{1}{|\Omega_x|} \int_{\Gamma_x} B(x,s,\vec{\nabla}_y U) \cdot \nu \, ds \ , \tag{1.2.d}$$

where $|\Omega_x|$ denotes the Lebesgue measure of Ω_x, and this contributes to the *cell storage*.

We discuss here an abstract system of evolution equations in Hilbert space in the form

$$\begin{aligned} &\frac{du}{dt} + Au + \frac{1}{\varepsilon}\alpha' C(\alpha u - \beta U) \ni f \quad \text{in} \quad H \\ &\frac{dU}{dt} + \mathcal{B}U + \frac{1}{\varepsilon}\beta' C(\beta U - \alpha u) = \mathcal{F} \text{ in } \ \mathcal{H} \end{aligned} \tag{1.3}$$

which was motivated by the preceding flow problems. Even for these examples our existence and convergence results are new. This is the first treatment of such problems which includes a *general* m*-accretive operator*; the second component is a monotone operator (or system of such operators) from a Banach space to its dual. The convergence to the system with matched boundary conditions as $\varepsilon \to 0$ was proved previously for a nonlinear case only in [9] for a similar problem. The abstract setting is closely related to classical results on the perturbation of m-accretive operators. We also mention below a rather broad variety of additional examples to suggest additional applications of the abstract system and to illustrate the limitations and possible extensions of our results.

Our plan is as follows. In Section 2 we develop our main results for the abstract evolution system (1.3). The well-posedness results for the system follow by showing that the corresponding stationary problem describes the generator of a semigroup of nonlinear contractions in $H \times \mathcal{H}$. With an additional assumption on the range of the operator α, we show that the solutions of (1.3) converge to the solution of a corresponding limiting problem; this is the "matched model" for the flow problems above. Thereafter we present in the successive following sections a collection of examples to which our preceding results directly apply. These include various dual porosity models and examples to illustrate the appropriateness of our additional hypothesis to ensure the convergence to the matched model.

2. The System.

Let H and $\mathcal{H}$ be Hilbert spaces, each of which is to be identified with its dual. Suppose $\mathcal{V}$ and $\mathcal{T}$ are Banach spaces with duals denoted respectively by $\mathcal{V}'$ and $\mathcal{T}'$, and let $\mathcal{V}$ be dense and continuously imbedded in $\mathcal{H}$ so we have $\mathcal{V} \subset \mathcal{H} \subset \mathcal{V}'$. We assume

A_1 • A is m-accretive on H,

A_2 • $\mathcal{B} : \mathcal{V} \to \mathcal{V}'$ is continuous, monotone, and bounded-coercive:

$\{\langle \mathcal{B}\Phi, \Phi\rangle + |\Phi|^2_{\mathcal{H}}\}$ bounded implies that $\|\Phi\|_{\mathcal{V}}$ and $\|\mathcal{B}\Phi\|_{\mathcal{V}'}$ are bounded,

and

A_3 • $C \in \mathrm{L}(\mathcal{T}, \mathcal{T}')$ is symmetric and monotone.

It follows that $\langle C\cdot, \cdot\rangle$ is a semi-scalar-product on $\mathcal{T}$; denote the kernel of C in $\mathcal{T}$ by $\ker(C)$, the annihilator of $\ker(C)$ in $\mathcal{T}'$ by $\ker(C)^{\perp}$, and the completion of $\mathcal{T}$ with this semi-scalar-product by T. The continuous extension of C will be denoted likewise; it gives the semi-scalar-product $(\varphi, \psi)_T = C\varphi(\psi)$, and C maps T *onto* its dual $T' \hookrightarrow \mathcal{T}'$. Finally, we assume given the pair of operators

A_4 • $\alpha \in \mathrm{L}(H, T)$, $\beta \in \mathrm{L}(\mathcal{V}, \mathcal{T})$, with β a surjection onto $\mathcal{T}$.

The corresponding continuous duals are denoted by $\alpha' : T' \to H$ and $\beta' : \mathcal{T}' \to \mathcal{V}'$; note that β' is one-to-one.

We shall consider the stationary system

$$u_\varepsilon + a_\varepsilon + \frac{1}{\varepsilon}\alpha' C(\alpha u_\varepsilon - \beta U_\varepsilon) = f \ , \quad a_\varepsilon \in A(u_\varepsilon) \ \text{ in } \ H \tag{2.1.a}$$

$$U_\varepsilon + \mathcal{B}(U_\varepsilon) + \frac{1}{\varepsilon}\beta' C(\beta U_\varepsilon - \alpha u_\varepsilon) = \mathcal{F} \ \text{ in } \ \mathcal{H} \tag{2.1.b}$$

with $f, \mathcal{F}$ given as above and $\varepsilon > 0$. This system will be shown to be well-posed and to approximate

$$u + a + \alpha'\tau = f \ , \qquad a \in A(u) \ \text{ in } \ H \ , \tag{2.2.a}$$

$$U + \mathcal{B}(U) - \beta'\tau = \mathcal{F} \ \text{ in } \ \mathcal{H} \ , \tag{2.2.b}$$

$$\alpha u - \beta U \in \ker(C) \ , \qquad \tau \in \ker(C)^\perp \ \text{ in } \ \mathcal{T}' \tag{2.2.c}$$

as $\varepsilon \to 0$ when, in addition, we have

A_5 • $\alpha \in \mathbb{L}(H, \mathcal{T})$.

This last assumption is necessary in order for (2.2.a) to be meaningful.

We begin by considering the system (2.1). Let $[f_j, \mathcal{F}_j] \in H \times \mathcal{H}$ for $j = 1, 2$ and denote a corresponding pair of solutions by $[u_j, U_j] \in H \times \mathcal{V}$. By subtracting the corresponding components of the system (2.1) and applying them to the respective components of the difference of the solutions, we obtain

$$|u_1 - u_2|_H^2 + \frac{1}{\varepsilon}C\alpha(u_1 - u_2)\big(\alpha(u_1 - u_2)\big) \le \frac{1}{\varepsilon}C\beta(U_1 - U_2)\big(\alpha(u_1 - u_2)\big) + |f_1 - f_2|_H |u_1 - u_2|_H$$

$$|U_1 - U_2|_{\mathcal{H}}^2 + \frac{1}{\varepsilon}C\beta(U_1 - U_2)\big(\beta(U_1 - U_2)\big) \le \frac{1}{\varepsilon}C\alpha(u_1 - u_2)\big(\beta(U_1 - U_2)\big) + |\mathcal{F}_1 - \mathcal{F}_2|_{\mathcal{H}} |U_1 - U_2|_{\mathcal{H}}$$

since A is accretive and $\mathcal{B}$ is monotone. By adding these estimates and applying Cauchy-Schwartz, there follows the fundamental estimate

$$|u_1 - u_2|_H^2 + |U_1 - U_2|_{\mathcal{H}}^2 \le |f_1 - f_2|_H^2 + |\mathcal{F}_1 - \mathcal{F}_2|_{\mathcal{H}}^2 \ .$$

This establishes the uniqueness of a solution and, moreover, that the system (2.1) determines an accretive operator in $H \times \mathcal{H}$. The same holds for the system (2.2), since the corresponding calculation leads to the term $\tau(\alpha(u_1 - u_2) - \beta(U_1 - U_2))$ which vanishes by (2.2.c).

The following surjectivity result shows that these systems lead to m-accretive operators on $H \times \mathcal{H}$.

Theorem 1. *Let the spaces and operators be given as above with A_1, A_2, A_3, A_4. Then for every $\varepsilon > 0$ there exists a unique solution $[u_\varepsilon, U_\varepsilon]$ of (2.1).*

If in addition A_5 holds, then there exists a unique solution $[u, U]$ of (2.2) and we have strong convergence $u_\varepsilon \to u$ in H, $U_\varepsilon \to U$ in $\mathcal{H}$ as $\varepsilon \to 0$.

Proof. Let $\omega \in T'$ and consider the equation

$$u + A(u) + \frac{1}{\varepsilon}\alpha'(C\alpha u - \omega) \ni f \ .$$

For a pair ω_j, $j = 1, 2$, and corresponding solutions u_j, we obtain as before

$$|u_1 - u_2|_H^2 + \frac{1}{\varepsilon}|\alpha(u_1 - u_2)|_T^2 \le \frac{1}{\varepsilon}(\omega_1 - \omega_2)\big(\alpha(u_1 - u_2)\big) \ ,$$

and this implies

$$2\varepsilon|u_1 - u_2|_H^2 + |\alpha(u_1 - u_2)|_T^2 \le |\omega_1 - \omega_2|_{T'}^2 \ .$$

By A_4 there is a constant $c > 0$ for which $|\alpha(u)|_T \le c|u|_H$, $u \in H$, so we have

$$\left(1 + \frac{2\varepsilon}{c^2}\right)^{1/2} |C\alpha(u_1 - u_2)|_{T'} \le |\omega_1 - \omega_2|_{T'} \ , \qquad \omega_1, \omega_2 \in T' \ .$$

This shows that the mapping $\omega \mapsto C\alpha u$ is a strict contraction on the Hilbert space T'. Proceeding similarly from the equation

$$U + \mathcal{B}(U) + \frac{1}{\varepsilon}\beta'(C\beta U - \omega) = \mathcal{F} \ , \qquad \omega \in T' \ ,$$

we obtain the estimate

$$|C\beta(U_1 - U_2)|_{T'} \le |\omega_1 - \omega_2|_{T'} \ , \qquad \omega_1, \omega_2 \in T'$$

so $\omega \mapsto C\beta U$ is a contraction. But a solution of (2.1) corresponds to a fixed-point of the composition of these two maps, and T' is a Hilbert space, so we have established the existence of a solution of (2.1).

Remark. Notice the asymmetry between $\omega \mapsto C\alpha u$ and $\omega \mapsto C\beta U$. The former is a strict contraction because α is continuous on H; the latter is only a contraction because β is not continuous on $\mathcal{H}$ but only on $\mathcal{V} \subset \mathcal{H}$.

The family of solutions $[u_\varepsilon, U_\varepsilon]$ of (2.1) with $\varepsilon > 0$ satisfies the a-priori estimate

$$|u_\varepsilon|_H^2 + |U_\varepsilon|_{\mathcal{H}^2}^2 + (a_\varepsilon, u_\varepsilon)_H + \mathcal{B}U_\varepsilon(U_\varepsilon) + \frac{1}{\varepsilon}|\alpha u_\varepsilon - \beta U_\varepsilon|_T^2 = (f, u_\varepsilon)_H + (\mathcal{F}, U_\varepsilon)_{\mathcal{H}} \ , \qquad \varepsilon > 0 \ .$$

It follows that $|u_\varepsilon|_H^2$ and $|U_\varepsilon|_{\mathcal{H}}^2 + \mathcal{B}U_\varepsilon(U_\varepsilon)$ are bounded so A_2 shows that $|U_\varepsilon|_{\mathcal{V}}$ and $|\mathcal{B}U_\varepsilon|_{\mathcal{V}'}$ are bounded. Then from (2.1.b) it follows that $\{\frac{1}{\varepsilon}C(\alpha u_\varepsilon - \beta U_\varepsilon)(\beta\Phi)\}$ is bounded for each $\Phi \in \mathcal{V}$, and thus by A_4 that $\{\frac{1}{\varepsilon}C(\alpha u_\varepsilon - \beta U_\varepsilon)\}$ is weakly, hence, strongly bounded in $\mathcal{T}'$. Finally, from A_5 and (2.1.a) it is clear that $|a_\varepsilon|_H$ is bounded. These remarks imply that there is a subsequence, which we denote momentarily by the same $\{u_\varepsilon, U_\varepsilon\}$, for which we have weak convergence $u_\varepsilon \rightharpoonup u$ and $a_\varepsilon \rightharpoonup a$ in H, $U_\varepsilon \rightharpoonup U$ in $\mathcal{V}$ and $\mathcal{B}U_\varepsilon \rightharpoonup b$ in $\mathcal{V}'$, and $\frac{1}{\varepsilon}C(\alpha u_\varepsilon - \beta U_\varepsilon) \rightharpoonup \tau$ in $\mathcal{T}'$. It follows by weak continuity of α, β and C that $\alpha u - \beta U \in \ker(C)$ and by symmetry of C that $\tau(\psi) = \lim \frac{1}{\varepsilon}C\psi(\alpha u_\varepsilon - \beta U_\varepsilon) = 0$ for $\psi \in \ker(C)$, so (2.2.c) holds.

In order to establish (2.2.a) and (2.2.b), it suffices to show $a \in A(u)$ and $b = \mathcal{B}(U)$. To this end we show

$$\limsup_{\varepsilon \to 0} \big((a_\varepsilon, u_\varepsilon - u)_H + \mathcal{B}U_\varepsilon(U_\varepsilon - U)\big) \le 0 \ . \tag{2.3}$$

By (2.1) this is equivalent to

$$\limsup_{\varepsilon \to 0} \big(|u|_H^2 - |u_\varepsilon|_H^2 + |U|_{\mathcal{H}}^2 - |U_\varepsilon|_{\mathcal{H}}^2 - \frac{1}{\varepsilon}C(\alpha u_\varepsilon - \beta U_\varepsilon)(\alpha u_\varepsilon - \beta U_\varepsilon - \alpha u + \beta U)\big) \le 0 \ .$$

Since C is monotone and (2.2.c) holds, we need only verify the above without the term with C, and this is equivalent to

$$\liminf_{\varepsilon \to 0} (|u_\varepsilon|_H^2 + |U_\varepsilon|_{\mathcal{H}}^2) \ge |u|_H^2 + |U|_{\mathcal{H}^2} \ .$$

But this is immediate from weak lower-semicontinuity of the norms, so (2.3) is proven. Next, for each $[\varphi, g] \in A$ and $\Phi \in \mathcal{V}$ we have

$$(a_\varepsilon - g, u_\varepsilon - \varphi)_H + (BU_\varepsilon - B\Phi)(U_\varepsilon - \Phi) \ge 0 \ ,$$

so using (2.3) to take the lim sup yields

$$(a - g, u - \varphi)_H + (b - \mathcal{B}\Phi)(U - \Phi) \ge 0 \ .$$

That is, we have

$$(a - g, u - \varphi)_H \ge 0 \ , \qquad [\varphi, g] \in A \ ,$$

and

$$(b - \mathcal{B}\Phi)(U - \Phi) \geq 0 \ , \qquad \Phi \in \mathcal{V} \ .$$

Since A is maximal accretive by A_1 and $\mathcal{B}$ is maximal monotone by A_2, we have $a \in A(u)$ and $b = \mathcal{B}U$ as desired, so (2.2) is established. Also there is at most one solution of (2.2), so the original sequence (and not just a special subsequence) converges weakly as indicated above.

Finally, we verify that the convergence of $[u_\varepsilon, U_\varepsilon]$ in $H \times \mathcal{H}$ is *strong*. Subtract (2.2.a) from (2.1.a) and apply to $u_\varepsilon - u$, subtract (2.2.b) from (2.1.b) and apply to $U_\varepsilon - U$; the sum of these operations gives

$$|u_\varepsilon - u|_H^2 + |U_\varepsilon - U|_{\mathcal{H}}^2 + \left(\frac{1}{\varepsilon}C(\alpha u_\varepsilon - \beta U_\varepsilon) - \tau\right)\big(\alpha(u_\varepsilon - u) - \beta(U_\varepsilon - U)\big)$$

$$= |u_\varepsilon - u|_H^2 + |U_\varepsilon - U|_{\mathcal{H}}^2 + \frac{1}{\varepsilon}|\alpha u_\varepsilon - \beta U_\varepsilon|_T^2 - \tau\big(\alpha(u_\varepsilon - u) - \beta(U_\varepsilon - U)\big) \leq 0 \ .$$

The last term converges to zero by A_5, so we obtain the desired strong convergence. ■

In some of our applications, the operator $\mathcal{B} : \mathcal{V} \to \mathcal{V}'$ is the realization of a family of elliptic boundary-value problems and β is a *trace* mapping onto boundary-values. To describe this situation abstractly we assume, in addition, that

A_6 • $\beta : \mathcal{V} \to \mathcal{T}$ is a strict homomorphism onto $\mathcal{T}$ and the kernel, $\mathcal{V}_0 \equiv \ker \beta$, is dense in $\mathcal{H}$.

Then we have the formal part $B_0 : \mathcal{V} \to \mathcal{V}_0'$ defined by restriction, $B_0\Phi = \mathcal{B}\Phi|_{\mathcal{V}_0}$, $\Phi \in \mathcal{V}$, and a natural domain $\mathcal{D} \equiv \{\Phi \in \mathcal{V} : \mathcal{B}_0\Phi \in \mathcal{H}\}$ for the Green's operator $\partial_{\mathcal{B}} : \mathcal{D} \to \mathcal{T}'$ for which

$$\mathcal{B}\Phi = B_0\Phi + \beta'\partial_{\mathcal{B}}(\Phi) \ , \qquad \Phi \in \mathcal{D} \ . \tag{2.4}$$

This is an abstract *Green's formula* which displays the operator $\mathcal{B}$ as a formal partial differential equation plus a natural boundary condition. ¿From (2.4) it follows that (2.1.b) and (2.2.b) have the respective equivalent forms

$$\begin{aligned} U_\varepsilon + B_0(U_\varepsilon) &= \mathcal{F} \ \text{ in } \ \mathcal{H} \ , \\ \partial_{\mathcal{B}}(U_\varepsilon) + \frac{1}{\varepsilon}C(\beta U_\varepsilon - \alpha u_\varepsilon) &= 0 \ \text{ in } \ \mathcal{T}' \ , \end{aligned} \tag{2.1.b$'$}$$

and

(2.2.b′)
$$U + B_0(U) = \mathcal{F} \ \text{ in } \ \mathcal{H}$$
$$\partial_{\mathcal{B}}(U) = \tau \ \text{ in } \ \mathcal{T}' \ .$$

In both cases, (2.1.a) and (2.2.a) are equivalent to

(2.5)
$$u + a + \alpha'(\partial_{\mathcal{B}} U) = f \ , \quad a \in A(u) \ \text{ in } \ H \ ,$$

Thus, the dual or natural boundary-values for U are a source-sink term in the equation for u.

The preceding results combine directly with the generation theory of semigroups of operators in Hilbert space to give the following [4], [5].

Theorem 2. *Let the spaces and operators be given as above with A_1, A_2, A_3, A_4. Let $f : [0,T] \to H$ and $\mathcal{F} : [0,T] \to \mathcal{H}$ be absolutely continuous and $\varepsilon > 0$, $u_0 \in \mathrm{dom}(A)$, $U_0 \in \mathcal{V}$ be given with $\mathcal{B}(U_0) + \frac{1}{\varepsilon}\beta' C(\beta U_0 - \alpha u_0) \in \mathcal{H}$. Then there exists a unique pair of Lipschitz continuous functions $u_\varepsilon : [0,T] \to H$, $U_\varepsilon : [0,T] \to \mathcal{H}$ which satisfy*

(2.6.a)
$$\frac{du_\varepsilon}{dt}(t) + a_\varepsilon(t) + \frac{1}{\varepsilon}\alpha' C\big(\alpha u_\varepsilon(t) - \beta U_\varepsilon(t)\big) = f(t) \ , \quad a_\varepsilon(t) \in A\big(u_\varepsilon(t)\big) \ ,$$

(2.6.b)
$$\frac{dU_\varepsilon}{dt}(t) + \mathcal{B}\big(U_\varepsilon(t)\big) + \frac{1}{\varepsilon}\beta' C\big(\beta U_\varepsilon(t) - \alpha u_\varepsilon(t)\big) = \mathcal{F}(t) \ , \quad \text{a.e. } \ t \in [0,T] \ ,$$

and

(2.7)
$$u_\varepsilon(0) = u_0 \quad , \quad U_\varepsilon(0) = U_0 \ .$$

Assume in addition that A_5 holds. Then there exists a unique pair of Lipschitz continuous functions $u : [0,T] \to H$, $U : [0,T] \to \mathcal{H}$ which satisfy

(2.8.a)
$$\frac{du}{dt}(t) + a(t) + \alpha'(\tau(t)) = f(t) \ , \quad a(t) \in A(u(t)) \ ,$$

(2.8.b)
$$\frac{dU}{dt}(t) + \mathcal{B}(U(t)) - \beta'(\tau(t)) = \mathcal{F}(t) \ ,$$

(2.8.c)
$$\alpha u(t) \beta U(t) \in \ker(C) \ , \quad \tau(t) \in \ker(C)^{\perp} \ , \ \text{a.e.} \ \ t \in [0,T] \ ,$$

(2.9) $$u(0) = u_0 \quad , \quad U(0) = U_0 \ ,$$

and we have strong convergence

$$u_\varepsilon \to u \ \ in \ \ C[0,T;H] \quad , \quad U_\varepsilon \to U \ \ in \ \ C[0,T;\mathcal{H}]$$

as $\varepsilon \to 0$.

3. A Diffusion-Convection Operator.

Let Ω be a bounded domain in $\mathbb{R}^n$ lying on one side of its smooth boundary, $\partial\Omega$, with unit outward normal $\vec{\nu}$. We shall construct an m-accretive operator, A, in $L^2(\Omega)$ which corresponds to a second-order divergence-form quasi-linear elliptic operator together with linear first-order terms to model diffusive and convective transport. This provides a generalization of the special *linear* case in which $Au = f$ is the realization in $L^2(\Omega)$ of the boundary-value problem

(3.1.a) $$-\Delta u - \vec{a}_2 \cdot \vec{\nabla} u = f \ \text{ in } \ \Omega \ ,$$

(3.1.b) $$u = 0 \ \text{ on } \ \Gamma_- \quad , \quad \frac{\partial u}{\partial \nu} + \vec{a}_2 \cdot \vec{\nu} u = 0 \ \text{ on } \Gamma_+$$

of mixed type, where Γ_- is a measurable subset of the boundary containing the *inflow* boundary, $\{s \in \partial\Omega : \vec{a}_2 \cdot \vec{\nu}(s) < 0\}$, and Γ_+ is its complement. This operator, or its generalization below, will be used in Section 4 and could be applied in Section 5 as well. The remainder of this Section consists of a precise but standard construction of a generalization of (3.1), and it is otherwise not necessary for an understanding of the following examples.

Let $H^1(\Omega)$ be the Hilbert space of (equivalence classes of) functions in $L^2(\Omega)$ with each generalized derivative $D_j u \in L^2(\Omega)$, $1 \le j \le n$. The norm is given by

$$\|u\|_{H^1(\Omega)} = \left(\sum_{j=0}^{n} \|D_j u\|_{L^2}^2 \right)^{1/2}$$

where D_0 denotes the identity. Let $\gamma : H^1(\Omega) \to L^2(\partial\Omega)$ be the trace map onto boundary values and denote its kernel by $H_0^1(\Omega)$. See [1] for information on these Sobolev spaces. We let $\vec{\nabla} = (D_1, D_2, \ldots, D_n)$ denote the *gradient* operator; the dual *divergence* is $\vec{\nabla}\cdot$.

We first construct a linear first-order *convective* operator, with constant coefficients for simplicity. Thus, let $\vec{a}_1, \vec{a}_2 \in \mathbb{R}^n$ be given and define $\Gamma_- = \{s \in \partial\Omega : (\vec{a}_1 + \vec{a}_2) \cdot \vec{\nu}(s) < 0\}$, $V = \{u \in H^1(\Omega) : \gamma u = 0 \text{ a.e. on } \Gamma_-\}$, and $\mathcal{A} : V \to V'$ by

$$\mathcal{A}u(v) = \int_\Omega \big((\vec{a}_1 \cdot \vec{\nabla} u)v + u\vec{a}_2 \cdot \vec{\nabla} v\big)\, dx \ , \qquad u, v \in V \ .$$

It follows easily that $\mathcal{A}$ is continuous and monotone; furthermore it can be written in the form

$$\mathcal{A}u(u) = \int_\Omega \big((\vec{a}_1 - \vec{a}_2) \cdot \vec{\nabla} u\big)v\, dx + \int_{\Gamma_+} \vec{a}_2 \cdot \vec{\nu}\, \gamma u\, \gamma v\, ds$$

to display its parts on Ω and on $\Gamma_+ \equiv \partial\Omega \sim \Gamma_-$, respectively.

The principle part of our operator is the subgradient of a convex lower semicontinuous function $\varphi : V \to \mathbb{R}_+$ of the form

$$\varphi(u) = \sum_{j=0}^{n} \int_\Omega \varphi_j\big(D_j u(x)\big)\, dx \ , \qquad u \in V \ ,$$

where each $\varphi_j : \mathbb{R} \to \mathbb{R}_+$ is convex and continuous with $\varphi_j(0) = 0$ for $0 \le j \le n$, and there are constants $c_0 > 0$ and C with

$$c_0(s^2 - 1) \le \varphi_j(s) \le C(s^2 + 1) \ , \qquad s \in \mathbb{R} \ , \ 0 \le j \le n \ .$$

Thus each term in φ is continuous on V and we can compute the subgradient $\partial\varphi$ termwise [9]. Finally, we obtain from [10] that the multi-valued operator $I + \partial\varphi + \mathcal{A} : V \to V'$ is onto.

Now we define A to be the restriction of $\partial\varphi + \mathcal{A}$ to $H = L^2(\Omega)$, i.e., $Au \ni f$ if $\partial\varphi(u) \ni f - \mathcal{A}u$ with $f \in H$, and this means there exist $g_j \in \partial\varphi_j(D_j u)$ in $L^2(\Omega)$ for $0 \le j \le n$ with

$$\sum_{j=0}^{n} \int_\Omega g_j D_j v\, dx + \mathcal{A}u(v) = \int_\Omega f v\, dx \ , \qquad v \in V \ .$$

This is equivalent to the pair of equations

$$\begin{aligned} &-\vec{\nabla} \cdot \vec{g} + g_0 + (\vec{a}_1 - \vec{a}_2) \cdot \vec{\nabla} u = f \ \text{ in } \ L^2(\Omega) \ , \\ &\vec{g} \cdot \vec{\nu} + \vec{a}_2 \cdot \vec{\nu}\, \gamma u = 0 \ \text{ in } \ L^2(\Gamma_+) \ , \end{aligned}$$

where $\vec{g} = (g_1, g_2, \ldots, g_n)$ and $\vec{g}\cdot\vec{\nu}$ has the appropriate generalized meaning in $H^{-1/2}(\Gamma_+)$, the dual of the trace space [15], [12]. Thus, $Au \ni f$ is equivalent to the boundary-value problem

$$u \in H^1(\Omega) : -\sum_{j=1}^{n} D_j \partial\varphi_j(D_j u) + \partial\varphi_0(u) + (\vec{a}_1 - \vec{a}_2)\cdot\vec{\nabla} u \ni f \text{ in } \Omega \ , \tag{3.2.a}$$

$$u = 0 \text{ on } \Gamma_- \ , \quad \sum_{j=1}^{n} \partial\varphi_j(D_j u)\cdot\nu_j + \vec{a}_2\cdot\vec{\nu} u \ni 0 \text{ on } \Gamma_+ \ , \tag{3.2.b}$$

in the precise form stated above. The monotone graphs $\partial\varphi_j$ permit nonlinear or even multi-valued flux, and the vectors $\vec{a}_1, \vec{a}_2$ give linear convection such as arises from gravitational influence [2]. Note that the linear case (3.1) is obtained from

$$\varphi_j(s) = \tfrac{1}{2}s^2 \ , \qquad 1 \le j \le n \ , \ \varphi_0 = 0 \ , \ \vec{a}_1 = \vec{0} \ .$$

Many other realizations of nonlinear boundary-value problems as m-accretive operators in $L^2(\Omega)$ are given in [3], [4] and their references; one can add convective terms to these examples by standard perturbation results.

4. Distributed Microstructure Models.

The example in this section is based on the work in [8]. We refer the reader to that paper and to [7], [14], [13] for a more detailed exposition and proofs of the claims made here. This example differs from the models above by permitting a far more general operator A.

In the microstructure model, a cell Ω_x and a boundary value problem is specified at each point x of a global region Ω. This development is made rigorous by the use of a continuous direct sum of Hilbert spaces as described below.

Let Ω be a bounded open subset of $\mathbb{R}^n$ and let $L^2(\Omega, L^2(\mathbb{R}^n))$ be the space of (equivalence classes of) Bochner square integrable functions from Ω into $L^2(\mathbb{R}^n)$. Let Q be a measurable subset of $\Omega\times\mathbb{R}^n$ and let Ω_x be the x-section $\Omega_x = \{y \in \mathbb{R}^n : (x, y) \in Q\}$. Identify $L^2(Q)$ as a subspace of $L^2(\Omega\times\mathbb{R}^n) \cong L^2(\Omega, L^2(\mathbb{R}^n))$ and each $L^2(\Omega_x)$ as a subspace of $L^2(\mathbb{R}^n)$ by zero extension. Thus we can identify

$$L^2(Q) \cong \Big\{U \in L^2\big(\Omega, L^2(\mathbb{R}^n)\big) : U(x) \in L^2(\Omega_x) \ , \quad \text{a.e.} \quad x \in \Omega\Big\} \ .$$

Denote the right side by $\mathcal{H} = L^2(\Omega, L^2(\Omega_x))$.

For each $x \in \Omega$, let $H^1(\Omega_x)$ be the Sobolev space of functions in $L^2(\Omega_x)$ whose first order (distributional) derivatives also lie in $L^2(\Omega_x)$. Let $H_0^1(\Omega_x)$ be the closure of C_0^∞ in $H^1(\Omega_x)$. We also define

$$\mathcal{V} \equiv \Big\{ U \in L^2\big(\Omega, L^2(\Omega_x)\big) : U(x) \in H^1(\Omega_x) \ , \text{ a.e. } x \in \Omega \\ \text{and } \int_\Omega \|U(x)\|^2_{H^1(\Omega_x)} \, dx < \infty \Big\} \ .$$

Let $\gamma_x : H^1(\Omega_x) \to L^2(\Gamma_x)$ be the *trace* maps from each cell to its boundary. It is well known that the image of γ_x, $H^{1/2}(\Gamma_x)$, is dense in $L^2(\Gamma_x)$. Let $\mathcal{T} = L^2(\Omega, H^{1/2}(\Gamma_x))$ and define $\beta : \mathcal{V} \to \mathcal{T}$ to be the *distributed trace*

$$\beta(U)(x,s) \equiv \gamma_x\big(U(x)\big)(s) \qquad \forall\ x \in \Omega\ ,\ \forall\ s \in \Gamma_x \ .$$

Clearly β satisfies A_4 and A_6.

In this example, C will be the identity on $T = L^2(\Omega, L^2(\Gamma_x))$. It will be convenient to weight the norm in the space T to include a scaling factor. Define a function w on Ω by

$$w(x) = \frac{1}{|\Omega_x|} \ .$$

For $U \in L^2(\Omega, L^2(\Gamma_x))$, we define the norm of U as follows:

$$\|U\|^2_{L^2(\Omega, L^2(\Gamma_x))} \equiv \int_\Omega \|U\|^2_{L^2(\Gamma_x)} \, w(x) \, dx \ .$$

We assume w is bounded away from 0 and is also bounded above so that the same elements belong to $L^2(\Omega, L^2(\Gamma_x))$ with this norm as with the standard norm (i.e., with $w \equiv 1$).

Let $\alpha : L^2(\Omega) \to L^2(\Omega, L^2(\Gamma_x))$ be defined by constant extension, i.e., $(\alpha u)(x,s) = u(x)$, $x \in \Omega$, $s \in \Gamma_x$. Clearly α satisfies A_4 and the stronger requirement A_5.

We construct the operator $\mathcal{B}$ on $L^2(\Omega, H^1(\Omega_x))$ as follows. Assume that we are given a function $\hat{B} : Q \times \mathbb{R}^n \to \mathbb{R}^n$. Assume $\hat{B}$ is measurable in its first two components and continuous in the third. Finally, assume that there exist functions $h_1 \in L^2(Q)$ and $h_0 \in L^1(Q)$ such that $\hat{B}$ satisfies for almost every $(x,y) \in Q$ and all $\xi, \eta \in \mathbb{R}^n$:

(4.1.a) $$|\hat{B}(x,y,\xi)| \le c|\xi| + h_1(x,y) \ ,$$

(4.1.b) $$\langle \hat{B}(x,y,\xi) - \hat{B}(x,y,\eta), \xi - \eta \rangle \ge 0 \ ,$$

(4.1.c) $$\hat{B}(x,y,\xi) \cdot \xi \ge c_0|\xi|^2 - h_0(x,y) \ .$$

For each $x \in \Omega$ define the operator $\mathcal{B}_x : H^1(\Omega_x) \to H^1(\Omega_x)'$ by

$$\mathcal{B}_x w(v) \equiv \int_{\Omega_x} \hat{B}\big(x, y, \vec{\nabla}_y w(y)\big) \cdot \vec{\nabla}_y v(y)\, dy \ , \qquad w, v \in H^1(\Omega_x) \ .$$

Define $B_x w$ to be the restriction of $\mathcal{B}_x w$ to $C_0^\infty(\Omega_x)$ so that

$$B_x w = -\vec{\nabla}_y \cdot \hat{B}(x, \cdot, \vec{\nabla}_y w)$$

in the sense of distributions on Ω_x for each $w \in H^1(\Omega_x)$. Define the corresponding distributed operator $\mathcal{B} : \mathcal{V} \to \mathcal{V}'$ by

$$\mathcal{B}U(\Phi) \equiv \int_\Omega \mathcal{B}_x\big(U(x)\big)\Phi(x)\, dx \ , \qquad U, \Phi \in L^2\big(\Omega, H^1(\Omega_x)\big) \ .$$

It can be shown that $\mathcal{B}$ satisfies A_2. The formal partial differential operator on Q is given by $(B_0 W)(x) = B_x W(x)$ in $\mathcal{V}_0' = L^2(\Omega, H^{-1}(\Omega_x))$ for each $W \in \mathcal{V} = L^2(\Omega, H^1(\Omega_x))$, and the boundary operator is determined by (2.4) to be an extension of

$$\partial_{\mathcal{B}}(\beta\Phi) = \hat{B}\big(x, y, \nabla_y \Phi(x, y)\big) \cdot \vec{\nu}_x(y) \ , \quad x \in \Omega \ , \ y \in \partial\Omega_x \ ,$$

on smooth functions Φ in $\mathcal{V}$, where $\vec{\nu}_x$ is the unit outward normal on $\partial\Omega_x$.

Let A be the realization in $L^2(\Omega)$ of the boundary-value problem (3.1). (Of course we could take any m-accretive operator in $L^2(\Omega)$, specifically the quasi-linear example given by (3.2).) Then from Theorem 2 we obtain the existence and uniqueness of a solution to the distributed microstructure system

(4.2.a) $$\frac{\partial u_\varepsilon}{\partial t} - \Delta u_\varepsilon - \vec{a}_2 \cdot \vec{\nabla} u_\varepsilon + \int_{\partial\Omega_x} \hat{B}(\vec{\nabla}_y U_\varepsilon) \cdot \vec{\nu}_x\, dy = f \text{ in } \Omega \times [0, T] \ ,$$

(4.2.b) $$u_\varepsilon = 0 \text{ on } \Gamma_- \ , \quad \frac{\partial u_\varepsilon}{\partial \nu} + \vec{a}_2 \cdot \vec{\nu} u_\varepsilon = 0 \text{ on } \Gamma_+ \times [0, T] \ ,$$

(4.2.c) $$\frac{\partial U_\varepsilon}{\partial t} - \vec{\nabla}_y \cdot \hat{B}(\vec{\nabla}_y U_\varepsilon) = \mathcal{F} \text{ in } Q \times [0, T] \ ,$$

(4.2.d) $$\hat{B}(\vec{\nabla}_y U_\varepsilon) \cdot \vec{\nu}_x + \frac{1}{\varepsilon}(\gamma_x U_\varepsilon - u_\varepsilon(x)) = 0 \text{ on } \Omega \times \partial\Omega_x \times [0, T]$$

with initial conditions (2.7) for $\varepsilon > 0$ and data $u_0, U_0, f, \mathcal{F}$. Also, as $\varepsilon \to 0$ this solution pair converges strongly to that of the matched microstructure system

(4.3.a) $$\frac{\partial u}{\partial t} - \Delta u - \vec{a}_2 \cdot \vec{\nabla} u + \int_{\partial\Omega_x} \hat{B}(\vec{\nabla}_y U) \cdot \vec{\nu}_x\, dy = f \text{ in } \Omega \times [0, T] \ ,$$

(4.3.b) $$u = 0 \text{ on } \Gamma_- \times [0,T] \ , \quad \frac{\partial u}{\partial \nu} + \vec{a}_2 \cdot \vec{\nu} u = 0 \text{ on } \Gamma_+ \times [0,T] \ ,$$

(4.3.c) $$\frac{\partial U}{\partial t} - \vec{\nabla}_y \cdot \hat{B}(\vec{\nabla}_y U) = \mathcal{F} \text{ in } Q \times [0,T] \ ,$$

(4.3.d) $$\gamma_x U = u(x) \ \text{ on } \ \Omega \times \partial\Omega_x \times [0,T]$$

with initial conditions (2.9). First order terms could similarly be added to (4.2.c) and (4.3.c) as before.

5. Sums of m-accretive operators.

We give some examples in which the convergence result does *not* hold. Consider the special case in which each of α, β and C is the identity, and β is the injection of $\mathcal{V}$ onto a dense subset of H; then $H = T = \mathcal{H}$, and (2.1) takes the form

(5.1.a) $$u_\varepsilon + A(u_\varepsilon) + \frac{1}{\varepsilon}(u_\varepsilon - U_\varepsilon) \ni f$$

(5.1.b) $$U_\varepsilon + \mathcal{B}(U_\varepsilon) + \frac{1}{\varepsilon}(U_\varepsilon - u_\varepsilon) = \mathcal{F}$$

in $H \times H$. The limiting case (2.2) as $\varepsilon \to 0$ is equivalent to the single equation

(5.2) $$2u + A(u) + \mathcal{B}(u) = f + \mathcal{F}$$

in H, when A_5 holds, and then it follows that the sum $A + \mathcal{B}$ is m-accretive on H. This is well known since A_5 implies in this situation that $\mathrm{dom}(\mathcal{B}) = \mathcal{V} = H$; see Corollary 2.7 of [5]. Without A_5 or some such additional hypothesis, the sum will certainly not necessarily be m-accretive, even in the very special case of a pair of linear symmetric regular-accretive operators.

Example 5.1. Let A be the $L^2(\Omega)$-realization of the Dirichlet problem: $A(u) = f$ in $L^2(\Omega)$ means

$$u \in H_0^1(\Omega) : \int_\Omega \vec{\nabla} u \cdot \vec{\nabla} \varphi = \int_\Omega f \varphi \text{ for all } \varphi \in H_0^1(\Omega) \ .$$

Let $\mathcal{V} = H^1(\Omega)$ and define $\mathcal{B} \in \mathrm{L}(\mathcal{V}, \mathcal{V}')$ by

$$\mathcal{B}u(\varphi) = \int_\Omega \vec{\nabla} u \cdot \vec{\nabla} \varphi \ , \qquad u, \varphi \in \mathcal{V} \ ;$$

then $\mathcal{B}(u) = f$ in $L^2(\Omega)$ is the weak form of the Neumann problem,

$$-\Delta u = f \text{ in } \Omega \ , \qquad \frac{\partial u}{\partial \nu} = 0 \text{ on } \partial\Omega \ .$$

Thus Example 5.1 is the *over-determined* problem of finding a solution satisfying both Dirichlet and Neumann boundary conditions.

Example 5.2. Let the spaces be given by $\mathcal{T} = \mathcal{V} = H_0^1(\Omega)$ and β be the identity. Assume we are given $\hat{B} : \Omega \times \mathbb{R}^n \to \mathbb{R}^n$ which is measurable in the first component, continuous in the second, and satisfies estimates of the form (4.1), but independent of y. Then define $\mathcal{B} : \mathcal{V} \to \mathcal{V}'$ by

$$\mathcal{B}w(v) = \int_\Omega \hat{B}(x, \vec{\nabla} w(x)) \cdot \vec{\nabla} v(x)\, dx \ , \qquad w, v \in \mathcal{V} \ . \tag{5.3}$$

Here we choose a different *pivot space* H by which to identify functionals with functions. Recall that $-\Delta$ is the Riesz isomorphism of the Hilbert space $V \equiv H_0^1(\Omega)$ with scalar product $(\vec{\nabla} u, \vec{\nabla} v)_{L^2(\Omega)}$ onto its dual, $V' \equiv H^{-1}(\Omega)$. This dual is also a Hilbert space, and the duality maps and scalar products are related by

$$g(\varphi) = (g, -\Delta\varphi)_{V'} = \big((-\Delta)^{-1} g, \varphi\big)_V \ , \qquad g \in V' \ , \ \varphi \in V \ .$$

Let A_0 be a maximal monotone graph in $\mathbb{R} \times \mathbb{R}$ and denote likewise the corresponding m-accretive operator on $L^2(\Omega)$: $g \in A_0(w)$ in $L^2(\Omega)$ if and only if

$$g, w \in L^2(\Omega) \ \text{ and } \ g(x) \in A_0(w(x)) \ , \ \text{ a.e. } \ x \in \Omega \ .$$

Define A on $H = V' = H^{-1}(\Omega)$ by $f \in A(u)$ in V' if and only if there is a $g \in A_0(u)$ in $L^2(\Omega)$ with

$$(g, \varphi)_{L^2(\Omega)} = g(\varphi) = (f, \varphi)_{V'} \ , \quad \varphi \in V \ .$$

¿From above we see that the realization in V' is given by $f = -\Delta g$ with $g \in V$, and so $A = -\Delta \circ A_0$ is the indicated composition. (This is actually a subgradient operator [4].) The coupling operator is chosen to be $Cg(h) = (g, h)_{V'}$, so $T = V'$, and we let α denote the identity on V'. The system (2.1) takes the form

$$u_\varepsilon \in L^2(\Omega) \ , \ u_\varepsilon - \Delta \circ A_0(u_\varepsilon) + \frac{1}{\varepsilon}(u_\varepsilon - U_\varepsilon) = f \text{ in } H^{-1}(\Omega) \tag{5.4.a}$$

$$U_\varepsilon \in H^1(\Omega) \ , \ U_\varepsilon + \mathcal{B}(U_\varepsilon) + \frac{1}{\varepsilon}(U_\varepsilon - u_\varepsilon) = \mathcal{F} \text{ in } H^{-1}(\Omega) \ , \tag{5.4.b}$$

in which the first component is a *semi-linear elliptic* equation and the second is of *quasi-linear* type. There is no common space on which both are accretive. The corresponding evolution problem (2.6) has first component of *porous-media* type, and the second component is a *degenerate parabolic* equation.

6. Evolution on the Boundary.

We present two examples of systems which contain an evolution equation on a boundary or interface. Here convergence holds *without* the assumption A_5.

Example 6.1. Let Ω be given as in Section 3; set $\mathcal{V} = H^1(\Omega)$ and define $\mathcal{B} : \mathcal{V} \to \mathcal{V}'$ by (5.3). Let $\beta : H^1(\Omega) \to L^2(\partial\Omega)$ be the trace γ with range $T = H^{1/2}(\partial\Omega)$ and kernel $H_0^1(\Omega)$. Then A_6 holds with $\mathcal{H} = L^2(\Omega)$, and we have the formal part of $\mathcal{B}$ on Ω

$$B_0(U) = -\vec{\nabla} \cdot \hat{B}(\cdot, \vec{\nabla} U) \in H^{-1}(\Omega) \tag{6.1}$$

and boundary operator

$$\partial_{\mathcal{B}} w = \hat{B}(\cdot, \vec{\nabla} w) \cdot \vec{\nu} \in H^{-1/2}(\partial\Omega) \tag{6.2}$$

determined by (2.4). Choose $H = L^2(\partial\Omega)$ and $Cw(\varphi) = (w, \varphi)_H$ on $H^{1/2}(\partial\Omega)$, so we have $T = H$ and $C = \alpha =$ identity.

Let A be *any* m-accretive operator on $L^2(\partial\Omega)$. Then (2.6) takes the form

$$\frac{\partial u_\varepsilon}{\partial t} + A(u_\varepsilon) + \frac{1}{\varepsilon}(u_\varepsilon - \beta U_\varepsilon) \ni f \ , \quad \text{and} \tag{6.3.a}$$

$$\partial_{\mathcal{B}}(U_\varepsilon) + \frac{1}{\varepsilon}(\beta U_\varepsilon - u_\varepsilon) = 0 \ \text{ in } \ L^2(\partial\Omega) \ , \tag{6.3.b}$$

$$\frac{\partial U_\varepsilon}{\partial t} + B_0(U_\varepsilon) = \mathcal{F} \ \text{ in } \ L^2(\Omega) \ , \quad 0 \le t \le T \ . \tag{6.3.c}$$

Although A_5 does *not* hold in this situation, one can show that the solution of (6.3) does converge to that of the system (2.8); this take the form

$$\frac{\partial}{\partial t}(\beta U) + A(\beta U) + \partial_{\mathcal{B}}(U) \ni f \text{ in } L^2(\partial\Omega) \ , \tag{6.4.a}$$

$$\frac{\partial}{\partial t}(U) + B_0(U) = \mathcal{F} \text{ in } L^2(\Omega) \ , \ 0 \le t \le T \ . \tag{6.4.b}$$

In typical applications to diffusion phenomena, (6.4.b) governs the diffusion in the region Ω and $\partial_{\mathcal{B}}(U)$ is the flux onto the boundary, $\partial\Omega$, where the diffusion is governed by (6.4.a). The operator A can be, for example, a (nonlinear) realization of the Laplace-Beltrami elliptic operator in tangential coordinates. Similar problems arise from standard approximations of flow in internal manifolds or cracks with very high permeability [6], [11].

Example 6.2. Let Ω and Ω_0 be disjoint bounded domains in $\mathbb{R}^n$, each lying on one side of its smooth boundary, and let Γ_- be a relatively open and connected subset of $\partial\Omega\cap\partial\Omega_0$. Denote the respective unit outward normals by $\vec{\nu}$ and $\vec{\nu}_0$. Set $\mathcal{V} = \{\Phi \in H^1(\Omega) : \gamma\Phi = 0$ a.e. on $\partial\Omega \sim \Gamma_-\}$ and let $\beta : \mathcal{V} \to L^2(\Gamma_-)$ be the trace with range $\mathcal{T} = H^{1/2}(\Gamma_-)$ in $\mathcal{V} = L^2(\Gamma_-)$. Define $\mathcal{B}$ on $\mathcal{V}$ by (5.3) so B_0 and $\partial_{\mathcal{B}}$ are given by (6.1) and (6.2) on Ω and Γ_-, respectively. Let C and α be identity on $H = L^2(\Gamma_-)$ as above.

Next we construct A on $H = L^2(\Gamma_-)$. Let $\vec{a}_2 \in \mathbb{R}^n$ be given with $\vec{a}_2 \cdot \vec{\nu}_0 < 0$ on Γ_- and set $\Gamma_+ = \partial\Omega_0 \sim \Gamma_0$. For each $u \in \mathcal{T}$ we can solve a non-homogeneous version of (3.1) to obtain a unique

(6.5.a) $$U_0 \in H^1(\Omega_0) : -\Delta U_0 - \vec{a}_2 \cdot \vec{\nabla} U_0 = 0 \text{ in } \Omega_0$$

(6.5.b) $$U_0 = u \text{ on } \Gamma_- \ , \quad \frac{\partial U_0}{\partial \nu_0} + \vec{a}_2 \cdot \vec{\nu}_0 U_0 = 0 \text{ on } \Gamma_+ \ .$$

Then we define $Au \in H$ by

$$(Au, \varphi)_H = \int_\Omega (\vec{\nabla} U_0 \cdot \vec{\nabla}\Phi_0 + U_0 \vec{a}_2 \cdot \vec{\nabla}\Phi_0)\, dx$$

for all $\Phi_0 \in H^1(\Omega_0)$ with $\Phi_0 = \varphi$ a.e. on Γ_-; note that such φ are dense in H. In view of (6.5.a) it follows that

$$Au = \frac{\partial U_0}{\partial \nu_0} + \vec{a}_2 \cdot \vec{\nu}_0 U_0 \text{ in } L^2(\Gamma_-) \ ;$$

this is an m-accretive operator arising from a bilinear form on $H^{1/2}(\Gamma_-)$. The system (2.6) has the form of (6.3), but on $L^2(\Gamma_-) \times L^2(\Omega)$ with $\gamma U = 0$ in $L^2(\Gamma_+)$, and (2.8) has the form (6.4) with the same modification. Furthermore, we can use the definition of A through (6.5) to write (2.6) in the equivalent form

(6.6.a) $$-\Delta U_0^\varepsilon - a_2 \cdot \nabla U_0^\varepsilon = 0 \text{ in } \Omega_0 \ ,$$

(6.6.b) $$\frac{\partial U_0^\varepsilon}{\partial \nu_0} + \vec{a}_2 \cdot \vec{\nu}_0 U_0^\varepsilon = 0 \text{ on } \Gamma_+ \ ,$$

(6.6.c) $$U_0^\varepsilon \equiv u_\varepsilon \ , \quad \partial_{\mathcal{B}} U_\varepsilon + \frac{1}{\varepsilon}(\beta U_\varepsilon - u_\varepsilon) = 0 \ , \text{ and}$$

(6.6.d) $$\frac{\partial u_\varepsilon}{\partial t} + A u_\varepsilon + \frac{1}{\varepsilon}(u_\varepsilon - \beta U_\varepsilon) = f \text{ on } \Gamma_- \ ,$$

(6.6.e) $$\frac{\partial U_\varepsilon}{\partial t} + B_0(U_\varepsilon) = \mathcal{F} \text{ in } \Omega \ ,$$

(6.6.f) $$U_\varepsilon = 0 \text{ on } \partial\Omega \sim \Gamma_- \ , \qquad 0 \le t \le T \ .$$

This is a degenerate-parabolic system consisting of the elliptic equation (6.6.a) and the parabolic equation (6.6.e) coupled by the boundary conditions (6.6.c) and (6.6.d). Although A_5 is not satisfied, the limiting case for $\varepsilon \to 0$ can be shown to exist as before in the form (2.8), that is, (6.6.c) becomes

$$U_0 = u = \beta U \text{ on } \Gamma_-$$

and (6.6.d) is replaced by

$$\frac{\partial u}{\partial t} + A(u) + \partial_{\mathcal{B}}(U) = f \text{ on } \Gamma_- \ .$$

Much of this carries over to the case of the more general operator A constructed as above from the boundary problem (3.2) on Ω_0.

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