Euler-Lagrange Equations

John Cook

March 17, 1994

Let J be a linear functional $C^2(I)$ given by

$$J(u) \equiv \int_I F(x, u, u')\, dx$$

where $F \in C^2(\mathbb{R}^3)$. We have

$$J(u + \lambda v) = \int_I F(x, u, u') + D_2 F(x, u, u')\lambda v + D_3 F(x, u, u')\lambda v' + \mathcal{O}(\lambda^2)\, dt.$$

If $v \in C_0^2(I)$ then integration by parts shows

$$\begin{aligned} \delta J(u; v) &\equiv \lim_{\lambda \to 0} \frac{J(u + \lambda v) - J(u)}{\lambda} \\ &= \int_I \left(D_2 F(x, u, u') - \frac{d}{dx} D_3 F(x, u, u') \right) v\, dt \end{aligned}$$

where $\delta J(u; v)$ is the Gâteaux variation of J at u in the direction of v. If $\delta J(u) = 0$ then we have the Euler-Lagrange equation

$$D_2 F - \frac{d}{dx} D_3 F = 0$$

or in more classical notation

$$\frac{\delta F}{\delta u} \equiv \frac{\partial F}{\partial u} - \frac{d}{dx} \frac{\partial F}{\partial u'} = 0.$$

Example

Consider the following example from mechanics. To match standard notation, we use t for x and $x(t)$ for $u(x)$. If we define

$$F(t, x, x') = \frac{m(x')^2}{2} - \frac{kx^2}{2},$$

then the corresponding Euler-Lagrange equation reduces to

$$mx'' + kx = 0.$$

Physically, F is the difference between kenetic and potential energy for a harmonic oscilator, the Lagrangian for the system.

Generalization

Suppose

$$J(u) \equiv \int_I F(x, u, u', u", \ldots, u^{(n)})\, dx.$$

Then

$$J(u+\lambda v) - J(u) = \int_I \left(\sum_{i=0}^{n} D_{i+2} F(x, u, u', \ldots, u^{(n)}) \lambda v^{(i)} + \mathcal{O}(\lambda^2) \right) dt,$$

or in other notation

$$\int_I \left(\sum_{i=0}^{n} \frac{\partial F}{\partial u^{(i)}} \lambda v^{(i)} + \mathcal{O}(\lambda^2) \right) dt.$$

If $v \in C_0^2(I)$ as before, integration by parts shows

$$\delta J(u; v) = \int_I \left(\sum_{i=0}^{n} (-1)^i \frac{d^i}{dx^i} \frac{\partial F}{\partial u^{(i)}} \right) v\, dt.$$

The Euler-Lagrange equation is

$$E(F) \equiv (-1)^i \frac{d^i}{dx^i} \frac{\partial F}{\partial u^{(i)}} = 0.$$

Extension to PDE's

Let $\Omega \subseteq \mathbb{R}^2$ and let J be given by

$$J(u) \equiv \int \int_\Omega F(x, y, u, u_x, u_y)\, dx\, dy.$$

If $v \in C_0^2(\Omega)$ then the Euler-Lagrange equation is

$$\frac{\partial F}{\partial u} - \frac{D}{Dx} \left(\frac{\partial F}{\partial u_x} \right) - \frac{D}{Dy} \left(\frac{\partial F}{\partial u_y} \right) = 0$$

where D/Dx is the total derivative

$$\frac{D\varphi}{Dx} \equiv \frac{\partial \varphi}{\partial x} + \frac{\partial \varphi}{\partial u} \frac{\partial u}{\partial x} + \frac{\partial \varphi}{\partial u_x} \frac{\partial^2 u}{\partial x^2}$$

and D/Dy is defined similarly.

Notes

- Several functionals can give rise to the same Euler-Lagrange equations.
- "If a differential equation can be derived from a variational principle, then the admittance of a Lie group is a necessary condition to find conservation laws by Noether's theorem." – Zwillinger p. 92

References:

E. Butkov, *Mathematical Physics*, Addison-Wesley 1968, chapter 13.

Daniel Zwillinger, *Handbook of Differential Equations*, Academic Press 1992, section I.20.