Notes on hypergeometric functions

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1 Definitions and notations

Define the kth rising power of a real number a by

$$a^{\overline{k}} = a(a+1)(a+2)\cdots(a+k-1).$$

Similarly, define the kth falling power of a by

$$a^{\underline{k}} = a(a-1)(a-2)\cdots(a-k+1).$$

We define

$$a^{\overline{0}} = a^{\underline{0}} = 1.$$

The quantity $a^{\overline{k}}$ is sometimes denoted by the "Pochammer symbol" $(a)_k$.

In calculations, it is often convenient to convert between rising powers, falling powers, and factorials. When a is not an integer, interpret $a!$ as $\Gamma(a+1)$ in the notes below.

$$\begin{aligned} a^{\overline{k}} &= (a+k-1)!/(a-1)! \\ a^{\underline{k}} &= a!/(a-k)! \\ (a+k)! &= a!(a+1)^{\overline{k}} \\ (a-k)! &= a!/a^{\underline{k}} \end{aligned}$$

A hypergeometric function is a function whose power series representation has the form

$$\sum_{k \geq 0} \frac{a_1^{\overline{k}} \ldots a_p^{\overline{k}}}{b_1^{\overline{k}} \ldots b_q^{\overline{k}}} \frac{z^k}{k!}.$$

We denote such a function by $F(a_1, \ldots, a_p; b_1, \ldots, b_q; z)$.

A hypergeometric function is called Gaussian if $p = 2$ and $q = 1$. This is the most common form and is often called *the* hypergeometric function. If $p = q = 1$ then the function is called a confluent hypergeometric function. Otherwise the function is called a generalized hypergeometric function.

One can show that a function is hypergeometric if and only if in the representation

$$f(z) = \sum_{k \geq 0} t_k z^k$$

the ratio

$$\frac{t_{k+1}}{t_k}$$

is a rational function of k. The parameters of the hypergeometric function are zeros and poles of this rational function.

2 Transformations

The function $F(a, b; c; z)$ satisfies seven identities known as linear transformations. These relationships are typographically complex and will only be summarized here. See Abramowitz and Stegun for full details.

These the linear transformations relate $F(a, b; c; z)$ to $F(a', b'; c'; z')$ where z is one of the following:

- z
- $1/z$
- $1 - z$
- $(z - 1)/z$
- $z/(1 - z)$
- $1/(z - 1)$

In terms of the parameters, the linear transformations relate $(a, b; c)$ to

- $(c - a, c - b; c)$
- $(a, c - b; c)$
- $(b, c - a; c)$
- $(a, b; a + b - c + 1)$ and $(c - a, c - b; c - a - b + 1)$
- $(a, 1 - c + a; 1 - b + a)$ and $(b, 1 - c + b; 1 - a + b)$

- $(a, c-b; a-b+1)$ and $(b, c-a; b-a+1)$
- $(a, a-c+1; a+b-c+1)$ and $(c-a, 1-a; c-a-b+1)$

There exist quadratic transformations if and only if out of the six numbers

$$\pm(1-c), \quad \pm(a-b), \quad \pm(a+b-c)$$

either two are equal or one equals 1/2. Again see Abramowitz and Stegun for details.

3 Hypergeometric forms of common functions

$$\begin{aligned}
e^z &= F(\cdot;\cdot;z) && (1)\\
(1-z)^a &= F(-a;\cdot;z) && (2)\\
(1+z)^{2a}+(1-z)^{2a} &= 2F(-a;\frac{1}{2}-a;\frac{1}{2};z^2) && (3)\\
(1+z)^{2a}-(1-z)^{2a} &= 4azF(\frac{1}{2}-a;1-a;\frac{3}{2};z^2) && (4)\\
\left(\frac{1}{2}+\frac{1}{2}(1-z)^{\frac{1}{2}}\right)^{1-2a} &= F(a,a-\frac{1}{2};2a;z) && (5)\\
(1+z)(1-z)^{-2a-1} &= F(a+1,2a;a;z) && (6)\\
\cos(z) &= F(\cdot;\frac{1}{2};-\frac{1}{4}z^2) && (7)\\
\sin(z) &= zF(\cdot;\frac{3}{2};-\frac{1}{4}z^2) && (8)\\
\cos(2az) &= F(-a,a;\frac{1}{2};\sin^2 z) && (9)\\
\sin(2az) &= 2a\sin zF(\frac{1}{2}+a,\frac{1}{2}-a;\frac{3}{2};\sin^2 z) && (10)\\
z\csc z &= F(\frac{1}{2},\frac{1}{2};\frac{3}{2};\sin^2 z) && (11)\\
\log(1-z) &= zF(1,1;2;z) && (12)\\
\log\left(\frac{1+z}{1-z}\right) &= 2zF(\frac{1}{2},1;\frac{3}{2};z^2) && (13)\\
\arcsin(z) &= zF(\frac{1}{2},\frac{1}{2};\frac{3}{2};z^2) && (14)\\
\arccos(z) &= F(\frac{1}{2},1;\frac{3}{2};-z^2) && (15)\\
\arctan(z) &= zF(\frac{1}{2},1;\frac{3}{2};-z^2) && (16)\\
I_x(a,b) &= \frac{x^a(1-x)^b}{aB(a,b)}F(1,a+b;a+1;x) && (17)
\end{aligned}$$

$$
\begin{aligned}
\gamma(a,z) &= \frac{z^a}{a}F(a;a+1;-z) = \frac{z^a}{a}e^{-z}F(a;a+1;z) && (18)\\
\mathrm{Erf}(z) &= zF(\frac{1}{2};\frac{3}{2};-z^2) = ze^{-z^2}F(1;\frac{3}{2};z^2) && (19)\\
P_n^{(\alpha,\beta)}(x) &= \frac{(\alpha+1)^{\overline{n}}}{n!}F(-n,n+\alpha+\beta+1;\alpha+1;\frac{1-x}{2}) && (20)\\
K(z) &= \frac{\pi}{2}F(\frac{1}{2},\frac{1}{2};1;z^2) && (21)\\
E(z) &= \frac{\pi}{2}F(-\frac{1}{2},\frac{1}{2};1;z^2) && (22)\\
J_\nu(z) &= \frac{(z/2)^\nu}{\Gamma(\nu+1)}F(\cdot;1+\nu;-\frac{z^2}{4}) && (23)\\
I_\nu(z) &= \frac{(z/2)^\nu}{\Gamma(\nu+1)}F(\cdot;1+\nu;\frac{z^2}{4}) && (24)
\end{aligned}
$$

In the above table,

$$
\begin{aligned}
I_x(a,b) &= \frac{1}{B(a,b)}\int_0^x t^{a-1}(1-t)^{b-1}\,dt \quad \text{(incomplete beta)}\\
\gamma(a,z) &= \int_0^z t^{a-1}e^{-t}\,dt \quad \text{(incomplete gamma)}\\
\mathrm{Erf}(z) &= \int_0^z e^{-t^2}\,dt \quad \text{(error function)}\\
K(z) &= \int_0^{\pi/2}(1-z^2\sin^2 t)^{-1/2}\,dt \quad \text{(complete elliptic integral 1st kind)}\\
E(z) &= \int_0^{\pi/2}(1-z^2\sin^2 t)^{1/2}\,dt \quad \text{(complete elliptic integral 2nd kind)}
\end{aligned}
$$

Also, $P_n^{(\alpha,\beta)}(x)$ is the nth Jacobi polynomial with parameters α and β.

4 References

"Generalized Hypergeometric Series" by W. N. Bailey, Cambridge (1935)

"Handbook of Mathematical Functions" by Abramowitz and Stegun (1964)

"The special functions and their approximations" by Yudell L. Luke v. 1 (1969)

"Concrete Mathematics" by Graham, Knuth, and Patashnik (1994)