Outline of Laplace Transforms

John D. Cook

August 2, 1994

1. Basic properties

   (a) Definition: $\mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)\, dt$. Denote $\mathcal{L}\{f(t)\}$ by $F(s)$ and $\mathcal{L}\{g(t)\}$ by $G(s)$.

   (b) Existence: A function f is of exponential order a if $\lim_{t\to\infty} |f(t)e^{-at}| = 0$ for some a. If $f(t)$ is piecewise continuous and of exponential order a, $F(s)$ exists for $s > a$.

   (c) $\mathcal{L}$ is linear: If a and b are constants, $\mathcal{L}\{af(t) + bg(t)\} = aF(s) + bG(s)$

   (d) $\mathcal{L}$ is invertible: If $F(s) = G(s)$, $f(t) = g(t)$. Thus it makes sense to say $\mathcal{L}^{-1}\{F(s)\} = f(t)$.

   (e) Decay: $\lim_{s\to\infty} F(s) = 0$.

2. Shifting and stretching

   (a) First Shifting Theorem: $\mathcal{L}\{e^{at} f(t)\} = F(s - a)$.

   (b) Second Shifting Theorem: $\mathcal{L}\{f(t - a)U(t - a)\} = e^{-as} F(s)$.

   (c) $f(t)$ has period T if there is a $T > 0$ such that $f(t + T) = f(t)$ for all t.

   (d) If f has period T, $\mathcal{L}\{f\} = \frac{1}{1 - e^{-sT}} \int_0^T e^{-st} f(t)\, dt$.

   (e) $\mathcal{L}\{f(ct)\} = \frac{1}{c} F\left(\frac{s}{c}\right)$.

3. Derivatives

   (a) Derivatives of Transforms: $\mathcal{L}\{t^n f(t)\} = (-1)^n \frac{d^n F(s)}{ds^n}$.

   (b) Transforms of Derivatives: $\mathcal{L}\{f^{(n)}(t)\} = s^n F(s) - s^{n-1} f(0) - s^{n-2} f'(0) - \cdots - f^{(n-1)}(0)$.

4. Convolution

   (a) Definition: $f \star g = \int_0^t f(\tau) g(t - \tau)\, d\tau$.

   (b) Theorem: $\mathcal{L}\{f \star g\} = \mathcal{L}\{f(t)\}\mathcal{L}\{g(t)\} = F(s)G(s)$.

5. Some Formulæ

(a) For $\alpha > -1$, $\mathcal{L}\{t^\alpha\} = \dfrac{\Gamma(\alpha+1)}{s^{\alpha+1}}$.

(b) For $x > 0$, $\Gamma(x+1) = x\Gamma(x)$.

(c) $\Gamma(n+1) = n!$ if n is a non-negative integer.

(d) $\Gamma(1/2) = \sqrt{\pi}$.

(e) $\mathcal{L}\{e^{at}\} = \dfrac{1}{s-a}$.

(f) $\mathcal{L}\{\sin kt\} = \dfrac{k}{s^2+k^2} \quad \mathcal{L}\{\cos kt\} = \dfrac{s}{s^2+k^2}$.

(g) $\mathcal{L}\{\sinh kt\} = \dfrac{k}{s^2-k^2} \quad \mathcal{L}\{\cosh kt\} = \dfrac{s}{s^2-k^2}$.

(h) $\mathcal{L}\{\delta(t-c)\} = e^{-sc}$.

http://www.johndcook.com/LaplaceTransforms.pdf