Introduction to Stokes' Equation

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Abstract

These notes are based on Roger Temam's book on the Navier-Stokes equations. They cover the well-posedness and regularity results for the stationary Stokes equation for a bounded domain.

1 Function Spaces

Let Ω be an open set in $\mathbb{R}^n$ with C^2 boundary Γ. Let $\mathcal{D}(\Omega)^n$ be the set of $\mathbb{R}^n$-valued smooth functions with compact support in Ω. Define

$$\mathcal{V} \equiv \{\vec{u} \in \mathcal{D}(\Omega)^n : \operatorname{div} \vec{u} = 0\}.$$

Let V be the closure of $\mathcal{V}$ in $H_0^1(\Omega)^n$ and let H be the closure of $\mathcal{V}$ in $L^2(\Omega)^n$. Note that for $\vec{u} = (u_1, \ldots, u_n)$ and $\vec{v} = (v_1, \ldots, v_n)$, the $L^2(\Omega)^n$ inner product is given by

$$(\vec{u}, \vec{v})_{L^2(\Omega)^n} = \sum_{i=1}^{n} (u_i, v_i)_{L^2(\Omega)}$$

and the $H^1(\Omega)^n$ inner product is given by

$$(\vec{u}, \vec{v})_{H^1(\Omega)^n} = \sum_{i=1}^{n} (u_i, v_i)_{H^1(\Omega)}.$$

Define

$$E(\Omega) \equiv \{\vec{u} \in L^2(\Omega)^n : \operatorname{div} \vec{u} \in L^2(\Omega)\}.$$

For $\vec{u}$ and $\vec{v}$ in $E(\Omega)$, define

$$(\vec{u}, \vec{v})_{E(\Omega)} = (\vec{u}, \vec{v})_{L^2(\Omega)^n} + (\operatorname{div} \vec{u}, \operatorname{div} \vec{v})_{L^2(\Omega)}.$$

Theorem 1 *$\mathcal{D}(\Omega)^n$ is dense in $E(\Omega)$.*

Proof The proof analogous to the proof that $\mathcal{D}(\Omega)$ is dense in $L^2(\Omega)$: for $\vec{u} \in E(\Omega)$, take the convolution of $\vec{u}$ with a mollifier φ_ε. To see that $\varphi_\varepsilon * \vec{u} \in E(\Omega)$, note that

$$\operatorname{div} (\varphi_\varepsilon * \vec{u}) = \varphi_\varepsilon * \operatorname{div} \vec{u}.$$

$\diamondsuit$

2 Trace Theorem

Let $\gamma_0 : H^1(\Omega) \to H^{1/2}(\Gamma)$ be the usual trace mapping and let $\ell_\Omega : H^{1/2}(\Gamma) \to H^1(\Omega)$ be defined by setting $\ell_\Omega(\varphi)$ equal to the solution to the Dirichlet problem on Ω with boundary data φ. Both are continuous linear maps. If we let $H^{-1/2}(\Gamma)$ denote the dual of $H^{1/2}(\Gamma)$ then

$$H^{1/2}(\Gamma) \hookrightarrow L^2(\Gamma) = L^2(\Gamma)' \hookrightarrow H^{-1/2}(\Gamma).$$

Theorem 2 *There exists a continuous linear operator $\gamma_\nu : E(\Omega) \to H^{-1/2}(\Gamma)$ such that $\gamma_\nu \vec{u} = \vec{u} \cdot \nu$ for every $u \in \mathcal{D}(\Omega)^n$ where ν is the unit outward normal. Also, the following generalization of Stokes' theorem holds: for every $\vec{u} \in E(\Omega)$ and $w \in H^1(\Omega)$,*

$$(\vec{u}, \operatorname{grad} w)_{L^2(\Omega)^n} + (\operatorname{div} \vec{u}, w)_{L^2(\Omega)} = \langle \gamma_\nu, \gamma_0 w \rangle.$$

Proof Define $X_u : H^{1/2} \to \mathbb{R}$ by

$$X_u \varphi = (\vec{u}, \operatorname{grad} w)_{L^2(\Omega)^n} + (\operatorname{div} \vec{u}, w)_{L^2(\Omega)}$$

for any w such that $\gamma_0 w = \varphi$. To see that X_u is well defined, suppose $\gamma_0 w_1 = \gamma_0 w_2$. Then $w = w_1 - w_2$ is in $H_0^1(\Omega)$ and hence is the limit of test functions w_ε. Then

$$(\vec{u}, \operatorname{grad} w_\varepsilon)_{L^2(\Omega)^n} + (\operatorname{div} \vec{u}, w_\varepsilon)_{L^2(\Omega)} = 0$$

by the classical Stokes theorem. Since γ_0 is continuous, the above equation holds for w as well as w_ε.

Let $w = \ell_\Omega(\varphi)$. Applying Cauchy-Schwarz to $[\vec{u}, \operatorname{div} \vec{u}]$ and $[\operatorname{grad} w, w]$ yields

$$|X_u \varphi| \le \|\vec{u}\|_E \|w\|_{H^1}$$

and so

$$|X_u \varphi| \le c \|\vec{u}\|_E \|\varphi\|_{H^{1/2}}$$

for some c by the continuity of ℓ_Ω. Thus X_u is a continuous linear functional on $H^{1/2}(\Gamma)$ and there exists $g \in H^{-1/2}(\Gamma)$ such that $X_u \varphi = \langle g, \varphi \rangle$. Define $\gamma_\nu \vec{u} = g$. To see that γ_ν behaves correctly on smooth functions, let $\vec{u}$ and w be smooth. Then

$$X_u(\gamma_0 w) = \int_\Omega \operatorname{div}(w\, \vec{u}) = \langle \vec{u} \cdot \nu, \gamma_0 w \rangle$$

by the classical Stokes theorem. Since the traces of smooth functions are dense in $H^{1/2}(\Gamma)$, the result holds by continuity. $\diamondsuit$

Theorem 3 *$\gamma_\nu : E(\Omega) \to H^{-1/2}(\Gamma)$ is onto.*

Proof Given $\psi \in H^{1/2}(\Gamma)$, let

$$\phi = \psi - \frac{\langle \psi, 1 \rangle}{\langle 1, 1 \rangle}.$$

Since $\langle \phi, 1 \rangle = 0$, there exists a unique solution to the Neumann problem

$$p \in H^1(\Omega) : \qquad \Delta p = 0, \quad \frac{\partial p}{\partial \nu} = \varphi$$

up to a constant. Thus $\operatorname{grad} p$ is unique. Let $\vec{u}_0$ be a C^1 function satisfying $\gamma_\nu = 1$. Then

$$\vec{u} \equiv \operatorname{grad} p + \frac{\langle \psi, 1 \rangle}{\langle 1, 1 \rangle} \vec{u}_0$$

satisfies $\gamma_\nu \vec{u} = \psi$. Also, the map $\psi \mapsto \vec{u}$ is continuous and linear. $\diamondsuit$

Let $E_0(\Omega)$ be the closure of $\mathcal{D}(\Omega)^n$ in $E(\Omega)$.

Theorem 4 $E_0(\Omega) = \ker \gamma_\nu$.

Proof The proof is analogous to the proof that $H_0^1 = \ker \gamma_0$. ◊

3 Characterization Theorems

3.1 Characterization of Gradients

Theorem 5 (De Rham) *A necessary and sufficient condition for a distribution f to be the gradient of another distribution is for f to vanish on $\mathcal{V}$, the set of divergence-free test functions.*

Assume from now on that Ω is bounded, unless otherwise stated.

Theorem 6 *If a distribution p has all its derivatives in $L^2(\Omega)$, or $H^{-1}(\Omega)$, then p is in $L^2(\Omega)$. In the first case,*

$$\|p\|_{L^2(\Omega)/\mathbb{R}} \leq c(\Omega)\|\mathrm{grad}\, p\|_{L^2(\Omega)}.$$

In the second,

$$\|p\|_{L^2(\Omega)/\mathbb{R}} \leq c(\Omega)\|\mathrm{grad}\, p\|_{H^{-1}(\Omega)}.$$

Note that

$$L^2(\Omega)/\mathbb{R} = \{u \in L^2(\Omega) : \int_\Omega u\, dx = 0\},$$

the orthogonal complement of the constant functions.

Corollary 1 *The divergence operator maps $H_0^1(\Omega)^n$ onto $L^2(\Omega)/\mathbb{R}$.*

Proof Let $A : L^2(\Omega) \to H^{-1}(\Omega)^n$ be the gradient operator. A is bounded linear operator and Theorem 6 shows that A is an isomorphism onto $Rg(A)$ and so $Rg(A)$ is closed. It follows that $(\ker A)^\perp = Rg(A^*)$. But $\ker A = \mathbb{R}$ and $A^* = -\mathrm{div}$. ◊

3.2 Characterization of Spaces

Theorem 7 *We may characterize H and its orthogonal complement in $L^2(\Omega)$ by*

$$H = \{\vec{u} \in L^2(\Omega)^n : \mathrm{div}\, \vec{u} = 0 \text{ and } \gamma_\nu \vec{u} = 0\}$$

and

$$H^\perp = \{\vec{u} \in L^2(\Omega)^n : \vec{u} = \mathrm{grad}\, p \text{ for some } p \in H^1(\Omega)\}.$$

Proof Characterization of $H^\perp$. "$\subseteq$": If $\vec{u}$ is perpendicular to H, is perpendicular to $\mathcal{V}$ and thus by De Rham's theorem, $\vec{u} = \mathrm{grad}\, p$ for some distribution p. Since $\vec{u} \in L^2(\Omega)$, Theorem 6 tells us $p \in L^2(\Omega)$ as well and so $p \in H^1(\Omega)$.

"$\supseteq$": $(\mathrm{grad}\, p, \vec{v})_{L^2(\Omega)^n} = -(p, \mathrm{div}\ \vec{v})_{L^2(\Omega)} = 0$ for all $\vec{v} \in \mathcal{V}$ and thus for all $\vec{v} \in V$.

Characterization of H. "$\subseteq$": If $\vec{u} \in H$, there exists a sequence $\vec{u}_n$ converging to $\vec{u}$ in $L^2(\Omega)$. $\mathrm{div}\, \vec{u}_n = 0$ for all n. Since distributional differentiation is continuous on $L^2(\Omega)$, $\mathrm{div}\, \vec{u} = 0$. This shows that $\vec{u}_n$ not only converges in $L^2(\Omega)$ but also in $E(\Omega)$. Since γ_ν is continuous on $E(\Omega)$ and $\gamma_\nu \vec{u}_n = 0$, $\gamma_\nu \vec{u} = 0$.

"$\supseteq$": H is a closed subspace of $L^2(\Omega)$ and thus any subspace properly containing it must contain an element of $H^\perp$. Suppose there exists a $\vec{u} \in H^\perp$ with $\mathrm{div}\, \vec{u} = 0$ and $\gamma_\nu \vec{u} = 0$. $\vec{u} = \mathrm{grad}\, p$ for some $p \in L^2(\Omega)$ and

$$\mathrm{div}\,(\mathrm{grad}\, p) = \Delta p = 0, \qquad \gamma_\nu \mathrm{grad}\, p = 0.$$

Thus p is a solution to the Neumann problem with zero data and so must be constant. But $\vec{u} = \mathrm{grad}\, p$ and thus $\vec{u} = 0$. ◊

Theorem 8 *$H^\perp$ can be split into the orthogonal spaces*

$$H_1 \equiv \{\vec{u} \in L^2(\Omega)^n : \vec{u} = \operatorname{grad} p \textit{ for some } p \in H^1(\Omega) \textit{ and } \Delta p = 0\},$$

and

$$H_2 \equiv \{\vec{u} \in L^2(\Omega)^n : \vec{u} = \operatorname{grad} p \textit{ for some } p \in H_0^1(\Omega)\}.$$

Theorem 9 *$V = \{\vec{u} \in H_0^1(\Omega)^n : \operatorname{div} \vec{u} = 0\}$.*

Proof "$\subseteq$": Follows from density of $\mathcal{V}$ and continuity of differentiation.

"$\supseteq$": Let W be the closed subspace of $H_0^1(\Omega)$ defined by the right side of the theorem statement. Suppose L is a functional defined on W which vanishes on V. Extend L to a functional on $H_0^1(\Omega)$. L vanishes on $\mathcal{V}$ and thus equals $\operatorname{grad} p$ for some $p \in L^2(\Omega)$ by Theorems 5 and 6. But then $L(\vec{v}) = -(p, \operatorname{div} \vec{v}) = 0$ for all $\vec{v} \in W$ and so $V = W$. ◇

We have assumed Ω is bounded. For general Ω's, the relationship between V and the divergence free members of $H_0^1(\Omega)^n$ was an open question as of 1985.

The relationship between the various spaces may be summarized by the following diagram.

$$\begin{array}{ccccccccc}
\mathcal{D}(\Omega)^n & & H_0^1(\Omega)^n & & L^2(\Omega)^n & & & & \\
\cup & & \cup & & \cup & & & & \\
\mathcal{V} & \subseteq & V & \subseteq & H & \subseteq & E_0 & \subseteq & E
\end{array}$$

4 Variational Formulation of Stokes' Equation

4.1 Homogeneous Problem

The strong form of the homogeneous steady-state Stokes problem is to find a function $\vec{u}$ representing velocity and a function p representing pressure such that

$$-v\Delta\vec{u} + \operatorname{grad} p = \vec{f} \in L^2(\Omega)^n \tag{1}$$

$$\operatorname{div} \vec{u} = 0 \in L^2(\Omega) \tag{2}$$

$$\gamma_0 \vec{u} = 0 \in H^{1/2}(\Gamma). \tag{3}$$

Here v represents kinematic viscosity, a positive constant. Also, the Laplacian is applied component-wise.

The divergence and boundary conditions on $\vec{u}$ are equivalent to asking that $\vec{u}$ be an element of $V \subseteq H$. Equation 1 says that $\vec{f} + v\Delta\vec{u}$ is an element of $H^\perp$. In this sense equation 1 and equations 2 and 3 are complementary.

Multiplication by a divergence-free test function $\vec{v} \in \mathcal{V}$ and integration by parts shows

$$v((\vec{u}, \vec{v})) = (\vec{f}, \vec{v})_{L^2(\Omega)^n} \tag{4}$$

for all $\vec{v} \in \mathcal{V}$ and thus for all $\vec{v} \in V$. Here $((\cdot,\cdot))$ is the principle part of the $H^1(\Omega)^n$ inner product.

Conversely, if $\vec{u} \in V$ satisfies equation 4 for all $\vec{v} \in \mathcal{V}$, then Theorems 5 and 6 show that

$$-v\Delta\vec{u} - \vec{f} = -\operatorname{grad} p$$

for some $p \in L^2(\Omega)$.

It is clear from the Lax-Milgram theorem that 4 is well posed even if Ω is only bounded in one direction, but our characterization of V depends on Ω being bounded. p is as uniquely determined as it could be: since only $\operatorname{grad} p$ appears in the equation, p could only possibly be unique up to a constant.

4.2 Non-Homogeneous Problem

Given $\vec{f} \in L^2(\Omega)^n$, $g \in L^2(\Omega)$, and $\vec{\varphi} \in H^{1/2}(\Gamma)^n$, we can solve

$$-v\Delta\vec{u} + \operatorname{grad} p = \vec{f} \in L^2(\Omega)^n \tag{5}$$

$$\operatorname{div} \vec{u} = g \in L^2(\Omega) \tag{6}$$

$$\gamma_0 \vec{u} = \vec{\varphi} \in H^{1/2}(\Gamma)^n \tag{7}$$

provided that

$$\int_\Omega g\, dx = \int_\Gamma \vec{\varphi} \cdot \nu\, ds. \tag{8}$$

Proof Pick $\vec{u}_0 \in H^1_0(\Omega)^n$ with $\gamma_0 \vec{u}_0 = \vec{\varphi}$. From the compatibility condition 8 and Stokes' formula,

$$\int_\Omega g - \operatorname{div} \vec{u}_0\, dx = 0.$$

Thus by Corollary 1, there exits $\vec{u}_1 \in H^1_0(\Omega)^n$ with $\operatorname{div} \vec{u}_1 = g - \operatorname{div} \vec{u}_0$. If we let $\vec{v} = \vec{u} - \vec{u}_0 - \vec{u}_1$ then the non-homogeneous Stokes problem for $\vec{u}$ reduces to the homogeneous Stokes problem for $\vec{v}$ with $\vec{f}$ replaced by $\vec{f} - v\Delta(\vec{u}_0 - \vec{u}_1)$. $\diamondsuit$

5 Regularity

Theorem 10 *Let $\Omega \subseteq \mathbb{R}^n$ be a bounded open with C^{m+2} boundary for a positive integer m. Let $1 < q < \infty$.Suppose that $\vec{u} \in W^{2,q}(\Omega)^n$ and $p \in W^{1,q}(\Omega)$ are solutions to the Stokes problem with data*

$$\vec{f} \in W^{m,q}(\Omega)^n,$$

$$\vec{g} \in W^{m+1,q}(\Omega)^n, \text{ and}$$

$$\vec{\varphi} \in W^{m+2-\frac{1}{q},q}(\Gamma)^n.$$

Then $\vec{u} \in W^{m+2,q}(\Omega)^n$ and $p \in W^{m+1,q}(\Omega)$. Also, there exists a constant $c(q, v, m, \Omega)$ such that

$$\|\vec{u}\| + \|p\| \leq c\{\|\vec{f}\| + \|\vec{g}\| + \|\vec{\varphi}\| + d\|\vec{u}\|_{L^q(\Omega)^n}\}$$

where $d = 0$ for $q \geq 2$ and $d = 1$ otherwise.

The unsubscripted norms in the above inequality are taken to be the strongest norms which make sense. In the case of p this means

$$\|p\|_{W^{m+1,q}(\Omega)/\mathbb{R}}$$

since p is only determined up to a constant.