Introduction to Stokes’ Equation

John D. Cook

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Abstract

These notes are based on Roger Temam’s book on the Navier-Stokes equations. They cover the well-posedness and regularity results for the stationary Stokes equation for a bounded domain.

1 Function Spaces

Let \( \Omega \) be an open set in \( \mathbb{R}^n \) with \( C^2 \) boundary \( \Gamma \). Let \( \mathcal{D}(\Omega)^n \) be the set of \( \mathbb{R}^n \)-valued smooth functions with compact support in \( \Omega \). Define

\[
\mathcal{V} \equiv \{ \vec{u} \in \mathcal{D}(\Omega)^n : \text{div} \vec{u} = 0 \}.
\]

Let \( \mathcal{V} \) be the closure of \( \mathcal{V} \) in \( \mathcal{H}^1_{0}(\Omega)^n \) and let \( \mathcal{H} \) be the closure of \( \mathcal{V} \) in \( \mathcal{L}^2(\Omega)^n \). Note that for \( \vec{u} = (u_1, \ldots, u_n) \) and \( \vec{v} = (v_1, \ldots, v_n) \), the \( \mathcal{L}^2(\Omega)^n \) inner product is given by

\[
(\vec{u}, \vec{v})_{\mathcal{L}^2(\Omega)^n} = \sum_{i=1}^{n} (u_i, v_i)_{\mathcal{L}^2(\Omega)}
\]

and the \( \mathcal{H}^1(\Omega)^n \) inner product is given by

\[
(\vec{u}, \vec{v})_{\mathcal{H}^1(\Omega)^n} = \sum_{i=1}^{n} (u_i, v_i)_{\mathcal{H}^1(\Omega)}.
\]

Define

\[
\mathcal{E}(\Omega) \equiv \{ \vec{u} \in \mathcal{L}^2(\Omega)^n : \text{div} \vec{u} \in \mathcal{L}^2(\Omega) \}.
\]

For \( \vec{u} \) and \( \vec{v} \) in \( \mathcal{E}(\Omega) \), define

\[
(\vec{u}, \vec{v})_{\mathcal{E}(\Omega)} = (\vec{u}, \vec{v})_{\mathcal{L}^2(\Omega)^n} + (\text{div} \vec{u}, \text{div} \vec{v})_{\mathcal{L}^2(\Omega)}.
\]

**Theorem 1** \( \mathcal{D}(\Omega)^n \) is dense in \( \mathcal{E}(\Omega) \).

**Proof** The proof analogous to the proof that \( \mathcal{D}(\Omega) \) is dense in \( \mathcal{L}^2(\Omega) \): for \( \vec{u} \in \mathcal{E}(\Omega) \), take the convolution of \( \vec{u} \) with a mollifier \( \varphi_\varepsilon \). To see that \( \varphi_\varepsilon \ast \vec{u} \in \mathcal{E}(\Omega) \), note that

\[
\text{div} (\varphi_\varepsilon \ast \vec{u}) = \varphi_\varepsilon \ast \text{div} \vec{u}.
\]

\( \diamond \)
2 Trace Theorem

Let \( \gamma_0 : H^1(\Omega) \to H^{1/2}(\Gamma) \) be the usual trace mapping and let \( \ell_\Omega : H^{1/2}(\Gamma) \to H^1(\Omega) \) be defined by setting \( \ell_\Omega(\varphi) \) equal to the solution to the Dirichlet problem on \( \Omega \) with boundary data \( \varphi \). Both are continuous linear maps. If we let \( H^{-1/2}(\Gamma) \) denote the dual of \( H^{1/2}(\Gamma) \) then

\[
H^{1/2}(\Gamma) \leftrightarrow L^2(\Gamma) = L^2(\Gamma)' \leftrightarrow H^{-1/2}(\Gamma).
\]

**Theorem 2** There exists a continuous linear operator \( \gamma_\nu : E(\Omega) \to H^{-1/2}(\Gamma) \) such that \( \gamma_\nu \bar{u} = \bar{u} \cdot \nu \) for every \( u \in \mathcal{D}(\Omega)^n \) where \( \nu \) is the unit outward normal. Also, the following generalization of Stokes’ theorem holds: for every \( \bar{u} \in E(\Omega) \) and \( w \in H^1(\Omega) \),

\[
\langle \bar{u}, \text{grad } w \rangle_{L^2(\Omega)^n} + \langle \text{div } \bar{u}, w \rangle_{L^2(\Omega)} = \langle \gamma_\nu, \gamma_0 w \rangle.
\]

**Proof** Define \( X_u : H^{1/2} \to \mathbb{R} \) by

\[
X_u\varphi = \langle \bar{u}, \text{grad } w \rangle_{L^2(\Omega)^n} + \langle \text{div } \bar{u}, w \rangle_{L^2(\Omega)}
\]

for any \( w \) such that \( \gamma_0 w = \varphi \). To see that \( X_u \) is well defined, suppose \( \gamma_0 w_1 = \gamma_0 w_2 \). Then \( w = w_1 - w_2 \) is in \( H^1_0(\Omega) \) and hence is the limit of test functions \( w_\varepsilon \). Then

\[
\langle \bar{u}, \text{grad } w_\varepsilon \rangle_{L^2(\Omega)^n} + \langle \text{div } \bar{u}, w_\varepsilon \rangle_{L^2(\Omega)} = 0
\]

by the classical Stokes theorem. Since \( \gamma_0 \) is continuous, the above equation holds for \( w \) as well as \( w_\varepsilon \).

Let \( w = \ell_\Omega(\varphi) \). Applying Cauchy-Schwarz to \( \langle \bar{u}, \text{div } \bar{u} \rangle \) and \( \langle \text{grad } w, w \rangle \) yields

\[
|X_u \varphi| \leq \|ar{u}\|_E\|w\|_{H^1}
\]

and so

\[
|X_u \varphi| \leq c\|ar{u}\|_E\|\varphi\|_{H^{1/2}}
\]

for some \( c \) by the continuity of \( \ell_\Omega \). Thus \( X_u \) is a continuous linear functional on \( H^{1/2}(\Gamma) \) and there exists \( g \in H^{-1/2}(\Gamma) \) such that \( X_u \varphi = \langle g, \varphi \rangle \). Define \( \gamma_\nu \bar{u} = g \). To see that \( \gamma_\nu \) behaves correctly on smooth functions, let \( \bar{u} \) and \( w \) be smooth. Then

\[
X_u(\gamma_0 w) = \int_\Omega \text{div } (w \bar{u}) = \langle \bar{u} \cdot \nu, \gamma_0 w \rangle
\]

by the classical Stokes theorem. Since the traces of smooth functions are dense in \( H^{1/2}(\Gamma) \), the result holds by continuity.

**Theorem 3** \( \gamma_\nu : E(\Omega) \to H^{-1/2}(\Gamma) \) is onto.

**Proof** Given \( \psi \in H^{1/2}(\Gamma) \), let

\[
\phi = \psi - \langle \psi, 1 \rangle \langle 1, 1 \rangle.
\]

Since \( \langle \phi, 1 \rangle = 0 \), there exists a unique solution to the Neumann problem

\[
p \in H^1(\Omega): \quad \Delta p = 0, \quad \frac{\partial p}{\partial \nu} = \varphi
\]

up to a constant. Thus \( \text{grad } p \) is unique. Let \( \bar{u}_0 \) be a \( C^1 \) function satisfying \( \gamma_\nu = 1 \). Then

\[
\bar{u} \equiv \text{grad } p + \langle \psi, 1 \rangle \bar{u}_0 \langle 1, 1 \rangle
\]

satisfies \( \gamma_\nu \bar{u} = \psi \). Also, the map \( \psi \mapsto \bar{u} \) is continuous and linear.

Let \( E_0(\Omega) \) be the closure of \( \mathcal{D}(\Omega)^n \) in \( E(\Omega) \).
Theorem 4 \( E_0(\Omega) = \text{ker} \gamma_\nu. \)

**Proof** The proof is analogous to the proof that \( H^1_0 = \text{ker} \gamma_\nu. \) \( \diamond\)

3 Characterization Theorems

3.1 Characterization of Gradients

**Theorem 5 (De Rham)** A necessary and sufficient condition for a distribution \( f \) to be the gradient of another distribution is for \( f \) to vanish on \( V \), the set of divergence-free test functions.

Assume from now on that \( \Omega \) is bounded, unless otherwise stated.

**Theorem 6** If a distribution \( p \) has all its derivatives in \( L^2(\Omega) \), or \( H^{-1}(\Omega) \), then \( p \) is in \( L^2(\Omega) \). In the first case,
\[
\|p\|_{L^2(\Omega)} \leq c(\Omega) \|\text{grad} p\|_{L^2(\Omega)}.
\]
In the second,
\[
\|p\|_{L^2(\Omega)} \leq c(\Omega) \|\text{grad} p\|_{H^{-1}(\Omega)}.
\]

Note that \( L^2(\Omega)/\mathbb{R} = \{ u \in L^2(\Omega) : \int_\Omega u \, dx = 0 \} \), the orthogonal complement of the constant functions.

**Corollary 1** The divergence operator maps \( H^1_0(\Omega) \) onto \( L^2(\Omega)/\mathbb{R} \).

**Proof** Let \( A : L^2(\Omega) \to H^{-1}(\Omega)^n \) be the gradient operator. \( A \) is bounded linear operator and Theorem 6 shows that \( A \) is an isomorphism onto \( \text{Rg}(A) \) and so \( \text{Rg}(A) \) is closed. It follows that \( (\ker A)^\perp = \text{Rg}(A^*) \).

Thus \( p \) is a solution to the Neumann problem with zero data and so must be constant. But \( \bar{u} = \text{grad} p \) and thus \( \bar{u} = 0. \) \( \diamond \)

3.2 Characterization of Spaces

**Theorem 7** We may characterize \( H \) and its orthogonal complement in \( L^2(\Omega) \) by
\[
H = \{ \bar{u} \in L^2(\Omega)^n : \text{div} \bar{u} = 0 \text{ and } \gamma_\nu \bar{u} = 0 \}
\]
and
\[
H^\perp = \{ \bar{u} \in L^2(\Omega)^n : \bar{u} = \text{grad} p \text{ for some } p \in H^1(\Omega) \}.
\]

**Proof** Characterization of \( H^\perp \). “\( \subseteq \)”: If \( \bar{u} \) is perpendicular to \( H \), is perpendicular to \( V \) and thus by De Rham’s theorem, \( \bar{u} = \text{grad} p \) for some distribution \( p \). Since \( \bar{u} \in L^2(\Omega) \), Theorem 6 tells us \( p \in L^2(\Omega) \) as well and so \( p \in H^1(\Omega) \).

“\( \supseteq \)”: \( (\text{grad} p, \bar{v})_{L^2(\Omega)^n} = -(p, \text{div} \bar{v})_{L^2(\Omega)} = 0 \) for all \( \bar{v} \in V \) and thus for all \( \bar{v} \in V \).

Characterization of \( H \). “\( \subseteq \)”: If \( \bar{u} \in H \), there exists a sequence \( \bar{u}_n \) converging to \( \bar{u} \) in \( L^2(\Omega) \). \( \text{div} \bar{u}_n = 0 \) for all \( n \). Since distributional differentiation is continuous on \( L^2(\Omega) \), \( \text{div} \bar{u} = 0 \). This shows that \( \bar{u}_n \) not only converges in \( L^2(\Omega) \) but also in \( E(\Omega) \). Since \( \gamma_\nu \) is continuous on \( E(\Omega) \) and \( \gamma_\nu \bar{u}_n = 0 \), \( \gamma_\nu \bar{u} = 0 \).

“\( \supseteq \)”: \( H \) is a closed subspace of \( L^2(\Omega) \) and thus any subspace properly containing it must contain an element of \( H^\perp \). Suppose there exists a \( \bar{u} \in H^\perp \) with \( \text{div} \bar{u} = 0 \) and \( \gamma_\nu \bar{u} = 0 \). \( \bar{u} = \text{grad} p \) for some \( p \in L^2(\Omega) \) and
\[
\text{div} (\text{grad} p) = \Delta p = 0, \quad \gamma_\nu \text{grad} p = 0.
\]
Thus \( p \) is a solution to the Neumann problem with zero data and so must be constant. But \( \bar{u} = \text{grad} p \) and thus \( \bar{u} = 0. \) \( \diamond \)
Theorem 8 \( H^\perp \) can be split into the orthogonal spaces
\[
H_1 \equiv \{ \vec{u} \in L^2(\Omega)^n : \vec{u} = \text{grad} \ p \text{ for some } p \in H^1(\Omega) \text{ and } \Delta p = 0 \},
\]
and
\[
H_2 \equiv \{ \vec{u} \in L^2(\Omega)^n : \vec{u} = \text{grad} \ p \text{ for some } p \in H^1_0(\Omega) \}.
\]

Theorem 9 \( V = \{ \vec{u} \in H^1(\Omega)^n : \text{div} \ \vec{u} = 0 \} \).

Proof "\( \subseteq \)" follows from density of \( V \) and continuity of differentiation.

"\( \supseteq \)" Let \( W \) be the closed subspace of \( H^1_0(\Omega) \) defined by the right side of the theorem statement. Suppose \( L \) is a functional defined on \( W \) which vanishes on \( V \). Extend \( L \) to a functional on \( H^1_0(\Omega) \). But then \( L(\vec{v}) = - (p, \text{div} \ \vec{v}) = 0 \) for all \( \vec{v} \in W \) and so \( V = W \).

We have assumed \( \Omega \) is bounded. For general \( \Omega \)'s, the relationship between \( V \) and the divergence free members of \( H^1_0(\Omega)^n \) was an open question as of 1985.

The relationship between the various spaces may be summarized by the following diagram.

\[
\begin{array}{cccc}
D(\Omega)^n & H^1_0(\Omega)^n & L^2(\Omega)^n & \\
\cup & \cup & \cup \\
V & \subseteq & V & \subseteq \\
& H & \subseteq & E_0 \subseteq E
\end{array}
\]

4 Variational Formulation of Stokes’ Equation

4.1 Homogeneous Problem

The strong form of the homogeneous steady-state Stokes problem is to find a function \( \vec{u} \) representing velocity and a function \( p \) representing pressure such that
\[
\begin{align*}
-v\Delta \vec{u} + \text{grad} \ p &= \vec{f} \in L^2(\Omega)^n \quad (1) \\
\text{div} \ \vec{u} &= 0 \in L^2(\Omega) \quad (2) \\
\gamma_0 \vec{u} &= 0 \in H^{1/2}(\Gamma). \quad (3)
\end{align*}
\]

Here \( \nu \) represents kinematic viscosity, a positive constant. Also, the Laplacian is applied component-wise.

The divergence and boundary conditions on \( \vec{u} \) are equivalent to asking that \( \vec{u} \) be an element of \( V \subseteq H \).

Equation 1 says that \( \vec{f} + v\Delta \vec{u} \) is an element of \( H^\perp \). In this sense equation 1 and equations 2 and 3 are complementary.

Multiplication by a divergence-free test function \( \vec{v} \in V \) and integration by parts shows
\[
v((\vec{u}, \vec{v})) = (\vec{f}, \vec{v})_{L^2(\Omega)^n} \quad (4)
\]
for all \( \vec{v} \in V \) and thus for all \( \vec{v} \in V \). Here \( ((\cdot, \cdot)) \) is the principle part of the \( H^1(\Omega)^n \) inner product.

Conversely, if \( \vec{u} \in V \) satisfies equation 4 for all \( \vec{v} \in V \), then Theorems 5 and 6 show that
\[
-v\Delta \vec{u} - \vec{f} = -\text{grad} \ p
\]
for some \( p \in L^2(\Omega) \).

It is clear from the Lax-Milgram theorem that 4 is well posed even if \( \Omega \) is only bounded in one direction, but our characterization of \( V \) depends on \( \Omega \) being bounded. \( p \) is as uniquely determined as it could be: since only \( \text{grad} \ p \) appears in the equation, \( p \) could only possibly be unique up to a constant.
4.2 Non-Homogeneous Problem

Given \( \vec{f} \in L^2(\Omega)^n, \ g \in L^2(\Omega), \) and \( \vec{\varphi} \in H^{1/2}(\Gamma)^n, \) we can solve

\[
-\nu \Delta \vec{u} + \text{grad} \ p = \vec{f} \in L^2(\Omega)^n \tag{5}
\]
\[
\text{div} \ \vec{u} = g \in L^2(\Omega) \tag{6}
\]
\[
\gamma_0 \vec{u} = \vec{\varphi} \in H^{1/2}(\Gamma)^n \tag{7}
\]

provided that

\[
\int_{\Omega} g \, dx = \int_{\Gamma} \vec{\varphi} \cdot \nu \, ds. \tag{8}
\]

**Proof** Pick \( \vec{u}_0 \in H^1_0(\Omega)^n \) with \( \gamma_0 \vec{u}_0 = \vec{\varphi}. \) From the compatibility condition 8 and Stokes’ formula,

\[
\int_{\Omega} g - \text{div} \ \vec{u}_0 \, dx = 0.
\]

Thus by Corollary 1, there exits \( \vec{u}_1 \in H^1_0(\Omega)^n \) with \( \text{div} \ \vec{u}_1 = g - \text{div} \ \vec{u}_0. \) If we let \( \vec{v} = \vec{u} - \vec{u}_0 - \vec{u}_1 \) then the non-homogeneous Stokes problem for \( \vec{u} \) reduces to the homogeneous Stokes problem for \( \vec{v} \) with \( \vec{f} \) replaced by \( \vec{f} - \nu \Delta (\vec{u}_0 - \vec{u}_1). \)

\[\Box\]

5 Regularity

**Theorem 10** Let \( \Omega \subseteq \mathbb{R}^n \) be a bounded open with \( C^{m+2} \) boundary for a positive integer \( m. \) Let \( 1 < q < \infty. \) Suppose that \( \vec{u} \in W^{2,q}(\Omega)^n \) and \( p \in W^{1,q}(\Omega) \) are solutions to the Stokes problem with data

\[
\vec{f} \in W^{m,q}(\Omega)^n, \qquad \vec{g} \in W^{m+1,q}(\Omega)^n, \quad \text{and} \quad \vec{\varphi} \in W^{m+2-1/q,q}(\Gamma)^n.
\]

Then \( \vec{u} \in W^{m+2,q}(\Omega)^n \) and \( p \in W^{m+1,q}(\Omega). \) Also, there exists a constant \( c(q,v,m,\Omega) \) such that

\[
\| \vec{u} \| + \| p \| \leq c\{ \| \vec{f} \| + \| \vec{g} \| + \| \vec{\varphi} \| + d\| \vec{u} \|_{L^q(\Omega)^n} \}
\]

where \( d = 0 \) for \( q \geq 2 \) and \( d = 1 \) otherwise.

The unsubscripted norms in the above inequality are taken to be the strongest norms which make sense. In the case of \( p \) this means

\[
\| p \|_{W^{m+1,q}(\Omega)/\mathbb{R}}
\]

since \( p \) is only determined up to a constant.