Suppose \( Y \sim \text{normal}(\theta, \sigma^2) \) and \( \text{a priori } \theta \sim \text{normal}(\mu, \tau^2) \). After observing \( y_1, y_2, \ldots, y_n \) the posterior distribution on \( \theta \) is normal with

\[
E(\theta \mid y_1 \ldots y_n) = \frac{\sigma^2/n}{\tau^2 + \sigma^2/n} \mu + \frac{\tau^2}{\tau^2 + \sigma^2/n} \bar{y}_{1:n}
\]

and

\[
\text{Var}(\theta \mid y_1 \ldots y_n) = \frac{\tau^2 \sigma^2/n}{\tau^2 + \sigma^2/n}
\]

If we were to observe future data \( y_{n+1}, y_{n+2}, \ldots, y_{n+m} \) the posterior distribution would again be normal with some mean \( \alpha \) and some variance \( \beta^2 \).

We can write \( \alpha \) as \( \alpha_0 + \alpha_1 \bar{y}_{n+1:n+m} \) where

\[
\alpha_0 = \frac{\sigma^2/(n + m)}{\tau^2 + \sigma^2/(n + m)} \mu + \frac{\tau^2}{\tau^2 + \sigma^2/(n + m)} \frac{n \bar{y}_{1:n}}{n + m}
\]

and

\[
\alpha_1 = \frac{\tau^2}{\tau^2 + \sigma^2/(n + m)} \frac{m}{n + m}
\]

Also,

\[
\beta = \frac{\tau^2 \sigma^2/(n + m)}{\tau^2 + \sigma^2/(n + m)}
\]

Now suppose we’re interested in whether a stopping rule will apply where a trial stops if

\[ P(\theta > c) < d \]

where the distribution on \( \theta \) is the posterior distribution after observing \( y_1, \ldots, y_{n+m} \). Now

\[ P(\theta > c) = 1 - P(\theta < c) = 1 - \Phi \left( \frac{\theta - \alpha}{\beta} \right) \]

so the trial will stop if and only if

\[ \Phi^{-1}(1 - d) < (c - \alpha)/\beta \]

or equivalently

\[ \bar{y}_{n+1:n+m} < k = \frac{c + \beta \Phi^{-1}(1 - d) - \alpha_0}{\alpha_1} \]
Note that everything on the right side above is known. That is, the trial will stop if and only if the sample average of the future observations is below a cutoff that does not depend on the data.

Therefore the predictive probability of the trial stopping is the predictive probability that \( \bar{y}_{n+1:n+m} \) is below a constant. The predictive distribution on \( \bar{y}_{n+1:n+m} \) is normal, and so the entire calculation reduces to computing the probability of a normal random variable being below a cutoff.

The mean of the predictive distribution on \( \bar{y}_{n+1:n+m} \) is the same as the posterior mean of \( \theta \) after observing \( y_1 \) through \( y_n \), i.e.

\[
\mu_p = \frac{\sigma^2/n}{\tau^2 + \sigma^2/n} \mu + \frac{\tau^2}{\tau^2 + \sigma^2/n} \bar{y}_{1:n}.
\]

The variance on the predictive distribution on \( \bar{y}_{n+1:n+m} \) is \( \sigma^2/m \) plus the variance on \( \theta \) given \( y_1 \) through \( y_n \), i.e.

\[
\sigma^2_p = \frac{\sigma^2}{m} + \frac{\tau^2 \sigma^2/n}{\tau^2 + \sigma^2/n}.
\]

Putting everything together, the probability of the trial stopping after \( m \) future observations is \( P(W < k) \) where \( W \) is normal with mean \( \mu_p \) and variance \( \sigma^2_p \). This equals or

\[
\Phi^{-1} \left( \frac{c + \beta \Phi^{-1}(1-d) - \alpha_0 - \alpha_1 \mu_p}{\alpha_1 \sigma_p} \right).
\]

The following R code implements the method described above.

```r
# xbar = current sample mean
# n = current number of observations
# m = future observations
# Trial stops if P(\theta | data > c) < d
# mu = prior mean on theta
# tau = prior variance on theta
# sigma = sampling variance
normalpp <- function(xbar, n, m, c, d, mu, tau, sigma) {
  temp <- (n+m)tau^2 + sigma^2
  alpha0 <- (mu*sigma^2 + m*xbar*tau^2) / temp
  alpha1 <- m*tau^2 / temp
  beta <- (tau*sigma)^2 / temp
  temp <- n*tau^2 + sigma^2
  mup <- mu*sigma^2 / temp + n*tau^2*xbar
  sigmap <- (sigma*tau)^2 / temp + sigma^2/m
  arg <- (c + beta*qnorm(1-d) - alpha0 - alpha1*mup)/(alpha1*sigmap)
  return( qnorm(arg) )
}
```

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