Orthogonal Polynomials and Gaussian Quadrature

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1 Introduction

Gaussian quadrature seems too good to be true. An n^{th} degree polynomial is determined by its values at $n+1$ points, so you might expect an integration rule based on $n+1$ points to integrate exactly polynomials of n degree. But Gaussian quadrature integrates exactly polynomials of degree $2n+1$ with $n+1$ integration points. By clever selection of the integration points and weights, you can accomplish about twice as much.

Despite its mysterious effectiveness, the properties of Gaussian quadrature are easy to prove. In fact, it's just as easy to prove the same results for a general class of integration schemes.

2 Orthogonal polynomials

Let $\{p_m(x)\}$ be a set of polynomials orthogonal over (a,b) with respect to the weight $w(x) > 0$. That is, $p_m(x)$ is an m^{th} degree polynomial and

$$\int_a^b p_m(x)\, p_n(x)\, w(x)\, dx = 0$$

unless $m = n$.

Theorem 1 *All the zeros of the $p_m(x)$ are real, have multiplicity 1, and lie in (a,b).*

Proof Let $a_1 \ldots a_n$ be the places where p_m changes signs in (a,b). Conceivably n could be zero, but we will show that in fact $n = m$. We know $n \leq m$ since $p_m(x)$ has at most m real zeros.

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The product $(x-a_1)(x-a_2)\cdots(x-a_n)p_m(x)$ never changes signs in (a,b). Since p_m is orthogonal to any polynomial of degree less than m,

$$\int_a^b (x-a_1)(x-a_2)\cdots(x-a_n)\, p_m(x)\, w(x)\, dx = 0$$

unless $m = n$. But the integrand is not zero and never changes signs. Therefore it must be the case that $m = n$. Since p_m changes signs m times we know that all the roots of p_m are isolated. ♢

3 Quadrature

We may develop a Gaussian quadrature method based on the polynomials p_m as follows. Let x_0, x_1, …, x_n be the roots of p_{n+1}. Let ℓ_i the the ith Lagrange interpolating polynomial for these roots, *i.e.* ℓ_i is the unique polynomial of degree $\leq n$ with $\ell_i(x_j) = \delta_{ij}$. Then

$$\int_a^b f(x)w(x)\, dx \approx \sum_{i=0}^{n} w_i f(x_i) \tag{1}$$

where the weights w_i are given by

$$w_i = \int_a^b \ell_i(x)w(x).$$

Theorem 2 *The approximation (1) above is exact if f is a polynomial of degree $\leq 2n+1$.*

Proof Supppose f is a polynomial of degree $\leq 2n+1$. Then $f = p_{n+1}q + r$ where q and r are polynomials of degree $\leq n$. Since p_{n+1} is orthogonal to all polynomials of degree $\leq n$,

$$\int_a^b f(x)w(x)\, dx = \int_a^b p_{n+1}(x)q(x)w(x)\, dx + \int_a^b r(x)w(x)\, dx = \int_a^b r(x)w(x)\, dx.$$

Now $r(x) = \sum_{i=0}^{n} r(x_i)\ell_i(x)$ and $f(x_i) = p_{n+1}(x_i)q(x_i) + r(x_i) = r(x_i)$ and so

$$\int_a^b r(x)w(x)\, dx = \sum_{i=0}^{n} f(x_i) \int_a^b \ell_i(x)w(x)\, dx = \sum_{i=0}^{n} w_i f(x_i).$$

♢

The most common Gaussian quadrature formula is the special case of $(a,b) = (-1,1)$ and $w(x) = 1$. In this case, the orthogonal polynomials are called Legendre polynomials.

4 Generalizations

For each family of orthogonal polynomials there is a corresponding integration rules. The table below gives the most common orthogonal polynomials and their names.

Name	Interval	Weight
Legendre	$(-1, 1)$	1
Laguerre	$(0, \infty)$	$x^\alpha e^{-x}$
Hermite	$(-\infty, \infty)$	e^{-x^2}
Chebyshev	$(-1, 1)$	$(1 - x^2)^{-1/2}$
Jacobi	$(-1, 1)$	$(1 - x)^\alpha (1 + x)^\beta$

5 Practical matters

Note that one can create a Newton-Cotes formula which evaluates the integrand at any $n + 1$ points in the interval (such as evenly spaced points), but such a method is only necessarily exact for polynomials of degree $\leq n$. But by picking the points to carefully, one can get a method which is exact for polynomials of degree $2n+1$. One gets much more accuracy for the same number of evaluations at the price of having to store the locations of the nodes. If the values of an integrand are given in the form of empirical data collected at points which we cannot choose, Gaussian quadrature is not appropriate.

Gaussian quadratures based on a weight $w(x)$ work remarkably well for functions that are approximately equal to a polynomial times the weight. However, it may take a change of variables to make this condition hold. For example, integrals arise in Bayesian statistics with the form $f(x)w(x)$ where $f(x)$ is very unlike a polynomial. For example, f may have horizontal asymptotes. Naive application of Gaussian quadrature based on weight $w(x)$ will not work well. Nevertheless, it may be possible to apply a change of variables that will make quadrature rules based on $w(x)$ work very well.