Picard's Theorem

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Theorem 1. *Let X be a Banach space and let $u_0 \in X$ be given. Let B be the ball of radius r centered at u_o. Suppose $f : [0,T] \times X \to X$ is continuous and satisfies the Lipschitz condition*

$$\|f(t,u) - f(t,v)\| \leq L\|u - v\|$$

for all $u, v \in B$ and $t \in [0,T]$. Suppose also that $\|f\| \leq M$ on $[0,T] \times B$. Let $c = \min(T, \frac{r}{M})$. Then the equation

$$u'(t) + f(t, u(t)) = 0$$

with initial condition $u(0) = u_0$ is well-posed in $C^1(0, c; X)$.

If X is a finite dimensional space, the Lipschitz condition is not necessary for existence, but it is necessary for uniqueness.

Proof. (Existence) The given differential equation is equivalent to the integral equation

$$u(t) = u_0 + \int_0^t f(s, u(s))\, ds. \tag{1}$$

This suggests the iteration procedure (Picard iteration) given by $u^0 = u_0$ and

$$u^{n+1} = u_0 + \int_0^t f(s, u^n(s))\, ds. \tag{2}$$

We show that the u^n converge uniformly to a solution u.

First of all,

$$\|u^1(t) - u^0(t)\| = \int_0^t f(s, u^0(s))\, ds \leq Mt.$$

This is where we need the hypothesis $t \leq \frac{r}{M}$, *i.e.* this condition is necessary to keep our iterates inside B. Next we notice

$$\begin{aligned}
\|u^2(t) - u^1(t)\| &\leq \int_0^t \|f(s, u^1(s)) - f(s, u^0(s))\| \, ds \\
&\leq \int_0^t L\|u^1(s) - u^0(s)\| \, ds \\
&\leq \int_0^t LMs \, ds = LM\frac{t^2}{2}.
\end{aligned}$$

In general,

$$\|u^{n+1}(t) - u^n(t)\| \leq \frac{M(Lt)^{n+1}}{L(n+1)!}.$$

Since

$$u^m(t) - u^n(t) = \sum_{i=1}^{m-1} (u^{i+1}(t) - u^i(t)),$$

it follows that

$$\|u^m(t) - u^n(t)\| \sum_{i=1}^{m-1} \frac{(Lc)^{i+1}}{(i+1)!}.$$

Thus, the sequence $\{u^n\}$ is uniformly Cauchy by comparison to the power series for e^{Lc}.

By the dominated convergence theorem, it follows from equation 2 that the limit u is a solution to equation 1. Here we only needed $f(t, x)$ to be continuous with respect to x and measurable with respect to t. □

The following simple version of Gronwall's inequality is necessary to show uniqueness and continuous dependence on initial conditions.

Lemma 1. *If* $w \in L^1[0, T]$ *and*

$$w(t) \leq C + L \int_0^t w(s) \, ds$$

then

$$w(t) \leq Ce^{Lt}.$$

Proof. Let $v(t) = C + L \int_0^t w(s) \, ds$. Since $v \geq w$ by hypothesis, we see that $e^{-Lt}v$ is non-increasing:

$$\frac{d}{dt} e^{-Lt} v(t) = e^{-Lt}(Lw - Lv) \leq 0.$$

So $e^{-Lt}v(t) \leq v(0) = C$ and $w(t) \leq v(t) \leq Ce^{Lt}$. □

The following establishes a growth estimate on solutions which also proves uniqueness and continuous dependence on initial conditions.

Proof. (Uniqueness) Suppose

$$u(t) = u_0 + \int_0^t f(s, u(s))\, ds$$

and

$$v(t) = v_0 + \int_0^t f(s, u(s))\, ds.$$

Let $w(t) = \|u(t) - v(t)\|$. The Lipschitz condition on f insures that

$$w(t) \leq w(0) + L \int_0^t w(s)\, ds.$$

□

The complex and real analytic analogs of Picard's theorem are also true: if f is complex (real) analytic, the solutions are complex (real) analytic. The basic idea of the proof is to use the real version of Picard's theorem on the real and imaginary parts. The integral operator in the existence proof preserves analyticity by Morera's theorem. See Garrett Birkhoff and Gian-Carlo Rota's text *Ordinary Differential Equations* for details.

In the case of X being a Hilbert space, the following theorem completes the proof that the problem

$$u'(t) + f(t, u(t)) = 0, \quad u(0) = u_0$$

is well-posed. Notice that this theorem independently establishes uniqueness and continuous dependence on initial conditions.

Theorem 2. *Let X be a Hilbert space. Let $g : [0,T] \times B \to X$ be a perturbation of f in the sense that*

$$\|f(t, w) - g(t, w)\| \leq \varepsilon$$

for every $t \in [0,T]$ and $w \in B$. If v is a solution to

$$v'(t) + g(t, v(t)) = 0$$

then for all $t \in [0,t]$,

$$\|u(t) - v(t)\| \leq \frac{\varepsilon}{L}(e^{Lt} - 1) + e^{Lt}\|u(0) - v(0)\|.$$

Proof. Define $\sigma(t) = \|u(t) - v(t)\|^2$. Then

$$\begin{aligned}
\frac{1}{2}\sigma'(t) &= \langle u(t) - v(t), u'(t) - v'(t)\rangle \\
&= \langle u(t) - v(t), f(t, u(t)) - g(t, v(t))\rangle \\
&\leq \langle u(t) - v(t), f(t, u(t)) - f(t, v(t))\rangle + \langle u(t) - v(t), f(t, v(t)) - g(t, v(t))\rangle \\
&\leq L\|u(t) - v(t)\|^2 + \varepsilon\|u(t) - v(t)\| \\
&= L\sigma + \varepsilon\sqrt{\sigma}.
\end{aligned}$$

A Gronwall-type inequality shows that

$$\sqrt{\sigma(t)} \leq \frac{\varepsilon}{L} e^{Lt} \sigma(t) + e^{Lt} \sqrt{\sigma(0)}.$$

$\square$

Notice that since this discussion has not involved any notion of accretiveness or monotonicity, the results hold on $[-T, T]$.

Picard's theorem has a number of important special cases; in fact, most of what is known about the general theory of ODE's is a special case or a variation of this theorem. For example, if $X = \mathbb{R}^n$ and $f(t, u(t)) = A(t)u(t) + g(t)$ where A is a matrix whose entries are continuous functions and g is a continuous function, then f is bounded on $[0, T] \times B$ for any bounded B. Thus any linear initial value problem is well-posed and solutions exist for all time. If we assume g and the components of A are L^∞ functions then the same result holds, except that the differential equation is satisfied only for almost every $t \in [0, T]$.