Separation of Convex Sets in Linear Topological Spaces

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Abstract

This paper discusses under what conditions two disjoint convex subsets of a linear topological space can be separated by a continuous linear functional. The equivalence of several forms of the Hahn-Banach theorem is proven. The separation problem is considered in linear topological spaces, locally convex linear topological spaces, Banach spaces, and finally finite dimensional Banach spaces. A number of examples are included to show the necessity of the hypotheses of various theorems.

1 Introduction

The purpose of this paper is to discuss under what conditions two disjoint convex subsets of a linear topological space can be separated by a continuous linear functional. Such separations are often described in geometric terms by discussing separation by closed hyperplanes. A hyperplane may be defined as a translate of the kernel of a non-trivial linear functional. Equivalently, a hyperplane is a translation of a subspace of co-dimension one. In a linear topological space a hyperplane is closed if and only if it is the kernel of a continuous linear functional. This leads to the geometric interpretation that two sets can be separated by a continuous linear functional if and only if it is possible to slip a closed hyperplane between them.

The Basic Separation Theorem was largely due to Mazur. He proved that a convex set with an interior point in a real normed space can be separated from any non-interior point. The more general statement given in this paper (Theorem 4) is due to Dieudonné. The proof given here is from Dunford and Schwartz (see [DS], p. 214). Tukey extended Mazur's work in [T].

All the separation results given rely on the Hahn-Banach Theorem (Theorem 1), which in turn requires the axiom of choice. The hypotheses concerning the linear space or its convex subsets allow the gauge function of some set to satisfy the hypothesis of the Hahn-Banach Theorem.

We discuss the Hahn-Banach Theorem in our section entitled Preliminary Results. Rather than give the standard proof, we consider a formally weaker version (Theorem 2), which suffices for all of our applications, and show that it is equivalent to what we call the Geometric Hahn-Banach Theorem (Theorem 3) which we prove directly. We also show that Theorem 1 follows from Theorem 2.

In our applications of the Hahn-Banach Theorem, general linear spaces will be discussed first. All linear spaces will be over the field of reals. Each section adds a new hypothesis concerning the linear space in question: linear topological spaces, locally convex linear topological spaces, Banach spaces, and finally finite dimensional Banach spaces, i.e. Banach spaces isomorphic to E^n, Euclidean n-space for some integer n.

A number of examples are included to show the necessity of the hypotheses of various theorems.

2 Notation

Throughout this paper, X will denote a real linear space. X' will denote the algebraic dual and X^* the continuous dual. A subspace of X will always refer to a linear subspace of X. A and B will be disjoint convex subsets of X. Other capital letters represent sets. $A - B$ will always mean the algebraic difference of A and B (not the set theoretic difference). Greek letters will represent real numbers; x, y, and z will represent elements of X.

The closure of a set C will be denoted $\overline{C}$. The linear subspace spanned by a set C will be notated $\langle C \rangle$. B_E denotes the closed unit ball of the normed linear space E. $B_\epsilon(x)$ denotes the open ball of radius ϵ centered at x. S_E is the closed unit sphere of E. $d(C, D)$ will be used for the distance between two sets in a normed space, $d(C, D) = \inf\{\| c - d \| : c \in C \text{ and } d \in D\}$.

3 Definitions

linear topological space A linear space X is called a linear topological space if X is a topological space in which the operations of vector addition and scalar multiplication are continuous. (Some authors require the topology to be Hausdorff.)

convex combination Let $\lambda_1 \ldots \lambda_n$ be non-negative reals whose sum is one and let $x_1 \ldots x_n$ be points of X. Then $\sum_{i=1}^{n} \lambda_i x_i$ is called a convex combination of the x_i's.

convex set $C \subseteq X$, is convex if every convex combination of two elements in C is also in C. Geometrically, the segment joining any two points in the set lies entirely in the set.

convex hull The convex hull of a subset $C \subseteq X$, denoted $\mathrm{co}(C)$, is the intersection of all convex sets containing C. Equivalently, $\mathrm{co}(C)$ is the set of all convex combinations of points in C.

closed convex hull The closed convex hull of a subset C of linear topological space X is the intersection of all closed convex sets containing C. Equivalently, it is $\overline{co(C)}$.

internal point A point p is an internal point of $C \subseteq X$ if for each point $x \in X$ there is a number $\epsilon > 0$ such that $p + ax \in C$ for all $\mid a \mid < \epsilon$.

affine span The affine span of $C \subseteq X$ is the set of all points in X that are linear combinations of points in C with real coefficients whose sum is one. Equivalently, the affine span of C is $\langle C + p \rangle$ where $p \in C$ is arbitrary.

relative interior In a linear topological space X, a point $p \in C$ is a relative interior point if it is an interior point of C in the topology induced by X on the affine span C.

locally convex linear topological space A linear topological space which has an open basis of convex sets.

gauge function Let $C \subseteq X$ be convex. The gauge function of C is defined by $g(x) = \inf\{\alpha > 0 : x \in \alpha C\}$. This may be extended real valued if 0 is not an algebraic interior point of C. Also called the Minkowski functional.

separate A linear functional $f \in X'$ is said to separate C and D if f is non-trivial and $\sup\{f(x) : x \in C\} \leq \inf\{f(x) : x \in D\}$. In this case we say C and D can be separated.

strictly separate f is said to strictly separate C and D if $\sup\{f(x) : x \in C\} < \inf\{f(x) : x \in D\}$.

4 Preliminary Results

Proposition 1 *If C and D are convex then $\alpha C + \beta D$ is convex for all real scalars α and β.*

Proof Assume x_1 and x_2 are in $\alpha C + \beta D$ with $x_i = \alpha c_i + \beta d_i$ for some $d_i \in D$ and some $c_i \in C$, $i = 1, 2$. Let σ and τ be two positive reals whose sum is one. Consider the convex combination

$$\sigma x_1 + \tau x_2 = \alpha(\sigma c_1 + \tau c_2) + \beta(\sigma d_1 + \tau d_2).$$

By convexity, the first term is in αC and the second is in βD. □

Proposition 2 *If C is a convex subset of a linear topological space, then $\overline{C}$ is convex.*

Proof Let C be convex. For fixed $\alpha \in [0,1]$, the function $f(x,y) = \alpha x + (1-\alpha)y$ is continuous. Thus

$$f[\overline{C} \times \overline{C}] = f[\overline{C \times C}] \subseteq \overline{f[C \times C]} \subseteq \overline{C}.$$

Thus every convex combination of two points in $\overline{C}$ is in $\overline{C}$ and hence $\overline{C}$ is convex. □

Proposition 3 *Let g be the gauge function of a convex set C which contains 0 as an internal point. Then*

1. *g is real valued,*
2. *$g(\gamma x) = \gamma g(x)$ for $\gamma \geq 0$,*
3. *$g(x) \leq$ if $x \in C$,*
4. *$g(x) \geq 1$ if $x \notin C$,*
5. *$g(x+y) \leq g(x) + g(y)$ for all x and y.*

Proof

1. g is real valued because C contains the origin as an internal point.
2. $g(\gamma x) = \inf\{\alpha > 0 : \gamma x \in \alpha C\} = \gamma \inf\{\alpha > 0 : x \in \alpha C\} = \gamma g(x)$.
3. If $x \in C$, $g(x) \leq 1$ by definition.
4. Suppose $g(x) < 1$. There exists some $\alpha < 1$ and $c \in C$ such that $x = \alpha c$. Since $0 \in C$, $x = \alpha c + (1-\alpha)0$ is a convex combination of points in C and thus is in C.
5. Suppose $\alpha > g(x)$, $\beta > g(y)$, and let $\gamma = \alpha + \beta$. $\frac{1}{\alpha}x$ and $\frac{1}{\beta}y$ are points in C from the definition of g.

$$\frac{1}{\alpha+\beta}\alpha(\frac{1}{\alpha}x) + \beta(\frac{1}{\beta}y) = \frac{1}{\gamma}(x+y)$$

   is a convex combination of points in C and thus in C. Thus $g(x+y) \leq \gamma$. Since $\alpha > g(x)$ and $\beta > g(y)$ are arbitrary, $g(x+y) \leq g(x) + g(y)$.

□

The main purpose of the above theorem is to show that the gauge function of a convex set with an internal point satisfies the hypothesis of the Analytic Hahn-Banach Theorem. The other parts will be used in the proofs of later theorems.

Theorem 1 (Analytic Hahn-Banach) *Let g be a real function on X satisfying $g(x+y) \leq g(x)+g(y)$ and $g(\gamma x)=\gamma g(x)$ for $\gamma \geq 0$ and $x, y \in X$. Let f be a linear functional on Y, a subspace of X, and suppose $f(x) \leq g(x)$ on Y. Then there exists a linear functional F extending f to X with $F(x) \leq g(x)$ for all $x \in X$.*

Proof The standard proof of this theorem is given in [DS] p. 62. □

There is a formally weaker version of Theorem 1 (Theorem 2 below) which suffices for all applications we require. We shall show that this formally weaker version is equivalent to the Geometric Hahn-Banach Theorem (Theorem 3) and give a direct proof of Theorem 3. This somewhat less familiar proof is of interest to us because it is stated in the language of separation. Our discussion is taken mostly from [RR]. Finally we shall show that Theorem 2 implies Theorem 1.

Theorem 2 *Let g be the gauge function of a convex subset of a linear space X which contains 0 as an internal point. Let f be a linear functional on Y, a subspace of X, and suppose $f(x) \leq g(x)$ on Y. Then there exists a functional F extending f to X with $F(x) \leq g(x)$ for all $x \in X$.*

Theorem 3 (Geometric Hahn-Banach) *Let X be a linear topological space. Let A be an open convex subset of X and M a subspace disjoint from A. Then there exists a closed hyperplane H containing M that is also disjoint from A.*

Proof By Zorn's lemma, there exists a maximal subspace N which contains M and is disjoint from A. We show that N is a closed hyperplane. Let D be the union of all λA where $\lambda > 0$. Let $C = N + D$. Clearly D is convex and thus by Proposition 1, C is convex. Suppose there is some $c \in C$ such that $-c \in C$. By convexity, $0 \in C$ but $0 \notin C$ since A and N are disjoint. Therefore C and $-C$ are disjoint. We will show that X is the union of C, $-C$ and N and that this implies N is a hyperplane.

Suppose $x \notin C$, $-C$, or N. Then $\langle N, x\rangle$ is disjoint from A, contradicting the maximality of N. Now suppose N is not a hyperplane. Then N has co-dimension greater than 1. There is a point $m \in C$ such that $\langle N, m\rangle \neq X$. Choose a point $b \in -C$, $b \notin \langle N, m\rangle$. The segment joining m and b is connected. Since A is open, D is open and thus C is open. Since C and $-C$ are disjoint open sets, there is a point $a = (1-\lambda)m + \lambda b \in N$. But this implies $b \in \langle N, m\rangle$. This contradiction implies N is a hyperplane. Hyperplanes are either closed or dense in X. Since A is open and disjoint from N, N is closed. □

Proposition 4 *Theorem 3 implies Theorem 2 .*

Proof Let g be the gauge function of a convex set containing 0 as an internal point. Let X have the linear topology induced by g and let $B = \{x : g(x) < 1\}$. Assume f is a non-trivial functional on a subspace Y of X satisfying $f(x) \leq g(x)$ for $x \in Y$. There exists some point $x_0 \in Y$ with $f(x_0) = 1$. Let $M = f^{-1}(0)$ and $A = x_0 - B$. A is open, convex and disjoint from the subspace M. Indeed, if $x_0 - b = m$ for some $b \in B$ and $m \in M$, then $f(x_0 - b) = f(m) = 0$ and so $f(b) = 1$. But $f(b) \leq g(b) < 1$. By the Geometric Hahn-Banach theorem, there is a closed hyperplane H containing M and disjoint from A. It follows that $\langle H, x_0\rangle = X$. Define $F(\lambda x_0 + h) = \lambda$ for real λ and $h \in H$. F extends f to all of X. We must prove $F(x) \leq g(x)$ for all $x \in X$.

Suppose there exists an x such that $F(x) > g(x)$. Then $g(\frac{x}{F(x)}) = \frac{g(x)}{F(x)} < 1$. Thus $\frac{x}{F(x)} \in B$ and $x_0 - \frac{x}{F(x)} \in A$. But $F(x_0 - \frac{x}{F(x)}) = 0$ and this contradicts the fact that H is disjoint from A. □

Proposition 5 *Theorem 2 implies Theorem 3.*

Proof Let a be a point in an open convex subset A of a linear topological space X and let M be a subspace of X disjoint from A. Let $C = A - a$ and let g be the gauge function of C. 0 is an interior point of C and hence (by Lemma 2 below) an internal point of C. Let $Y = \langle M, a\rangle$. Now every point in Y is of the form $m - \lambda a$, where $m \in M$. Let f be defined on Y by $f(m - \lambda a) = l$. $f(m - \lambda a) = \lambda \leq g(m - \lambda a)$ on Y. Indeed for $\lambda > 0$,

$$\frac{1}{\lambda}(m - \lambda a) = \frac{1}{\lambda}m - a \notin C = A - a$$

since A and M are disjoint. Extend f by Theorem 4 to a function F defined on X with $F(x) \leq g(x)$ for all x. Let $H = F^{-1}(0)$. Since F is zero on M, $M \subseteq H$. Also, for $b \in A$, $F(b) + 1 = F(b - a) \leq g(b - a) < 1$ and so $F(b) < 0$. Thus H and A are disjoint. Since A is open, the hyperplane H is closed. □

To complete our discussion of the Hahn-Banach Theorem we shall show that Theorem 2 implies Theorem 1. The converse is trivial.

Observation: Let g be a real valued function on a linear space X which is sublinear (*i.e.* g satisfies the hypothesis of Theorem 4) and let f be a linear functional on a subspace Y of X with $f(x) \leq g(x)$ for $x \in Y$. Then there exists a linear extension $\tilde{f}$ of f to all of X with $\tilde{f}(x) \leq g(x)$ for all $x \in X$ such that $g(x) \geq 0$. To see this, let $B = \{x \in X : g(x) \leq 1\}$. B is obviously convex and contains 0 as an internal point. Let p be the gauge function of B. An easy argument shows that $p(x) = g(x)$ if $g(x) \geq 0$. Since $p(x) \geq 0$ for all x, it follows that $f(x) \leq p(x)$ for all x. Thus the observation follows directly from Theorem 2.

Of course the observation is weaker than Theorem 1, but there is a cute trick we can use to deduce Theorem 1 from the observation. Let f, Y, g, and X be as in Theorem 1. Let F be the linear functional on $Y \oplus R$ given by $F(x,t) = f(x) + t$. Let G be defined on $X \oplus R$ by $G(x,t) = g(x) + t$. Clearly G is sublinear and $F(x,t) \leq G(x,t)$ for all $x \in Y$ and $t \in R$. By the observation, there is a linear extension $\tilde{F}$ of F to all of $X \oplus R$ with $\tilde{F}(x,t) \leq G(x,t)$ whenever $G(x,t) \geq 0$.

Now any linear functional on $X \oplus R$ is of the form $f(x) + \alpha t$. Since $\tilde{F}$ extends f, we see that $\alpha = 1$. Since $\tilde{F}(x,0) = f(x)$ for all $x \in Y$, $\tilde{f}$ is an extension of f to all of X. Fix $x \in X$. We claim that $\tilde{f}(x) \leq g(x)$. Indeed, choose $t \in R$ so that $G(x,t) = g(x) + t > 0$. Thus $\tilde{F}(x,t) = \tilde{f}(x) + t \leq g(x) + t$, hence $\tilde{f}(x) \leq g(x)$.

We conclude this section with an elementary observation.

Lemma 1 *A linear functional (strictly) separates arbitrary sets C and D iff it (strictly) separates $C-D$ and $\{0\}$.*

Proof $\sup f(x) : x \in C \ \leq\ \inf\{f(x) : x \in D\} \Leftrightarrow \sup\{f(x) : x \in C-D\} \ \leq\ 0 = f(0)$. Also, $\sup\{f(x) : x \in C\} < \inf\{f(x) : x \in D\} \Leftrightarrow \sup\{f(x) : x \in C-D\} < 0 = f(0)$. □

5 Linear Spaces

Theorem 4 (Basic Separation Theorem) *If A and B are disjoint convex sets and A has an internal point, then A and B can be separated.*

Proof Suppose x is an internal point of A. A and B can be separated if and only if $A-x$ and $B-x$ can be separated. Therefore, we may assume 0 is an internal point of A without loss of generality. Let $p \in B$. Now $-p$ is an internal point of $A-B$ and thus the origin is an internal point of the set $K = A-B+p$. Since A and B are disjoint, $A-B$ does not contain 0 and so K does not contain p.

Let g be the gauge function of K. $g(p) \ \geq\ 1$ since $p \notin K$. Define $f(p)$ to be $g(p)$ and extend linearly to all real multiples of p. $f(x) \ \leq\ g(x)$ on $\langle p \rangle$. Extend f to all of X by Hahn-Banach in such a way that $f(x) \ \leq\ g(x)$ for all x. Thus $f(x) \ \leq\ 1$ for all $x \in K$ and $f(p) \ \geq\ 1$. Therefore f separates K and p. But then f separates $A-B$ from 0 and by Lemma 1, f separates A and B. □

The hypothesis that one of the sets have an internal point cannot be removed without additional hypothesis. Goffman and Pedrick give the following counterexample (see [GP], p. 61).

Example 1: Let X be a linear space with a countably infinite Hamel basis $\{x_1, x_2, x_3 \ldots\}$. Let S be the set of all vectors whose representation as a linear combination of basis elements has a positive final coefficient. This set is convex and does not contain 0. Any functional that separates $\{0\}$ and S must be non-negative or non-positive on S. Let f be a linear functional on X which is non-negative on S. For every positive integer n and real ρ, $\rho x_n + x_{n+1} \in S$. Then $f(\rho x_n + x - n + 1) = \rho f(x_n) + f(x_{n+1}) \ \geq\ 0$. But this implies f vanishes on each basis element and thus f is the trivial functional. Thus no functional separates $\{0\}$ and S.

Strict separation of disjoint closed convex sets is not always possible even in the plane. Barbu's text (see [BP], p. 21) gives the following example of disjoint closed convex sets in the plane that cannot be strictly separated.

Example 2: Let $A = \{(x, y) : x \ \leq\ 0\}, B = \{(x, y) : xy \ \geq\ 1, x > 0, y > 0\}$. The hyperplane $x = 0$ separates A and B but does not strictly separate A and B.

6 Linear Topological Spaces

Lemma 2 *An interior point in a linear topological space is an internal point.*

Proof By translation, it suffices to prove that the origin is an internal point of any open set containing it. Suppose U is an open set containing 0. Select an arbitrary element y of X. Since scalar multiplication is continuous, there is an open set $(-\epsilon, \epsilon) \times V$ mapped into U where V is an open set from X containing y. In particular, ay is in U for every $\mid \alpha \mid < \epsilon$. □

Lemma 3 *If a linear functional on a linear topological space separates two sets, one of which has an interior point, that functional is continuous.*

Proof Let f be a linear functional separating C and D. Without loss of generality, we may assume 0 is an interior point of C. Then there exists g with $\sup\{f(x) : x \in C\} \leq \gamma \leq \inf\{f(x) : x \in D\}$. Let U be a neighborhood of the origin contained in C. Let V be the intersection of U and -U. V is a neighborhood of the origin. f restricted to V takes on values only in $[-\gamma, \gamma]$. A linear functional bounded in a neighborhood of the origin is continuous. □

Corollary 1 *Let A and B be disjoint convex sets of a linear topological space X. If A has non-empty interior, A and B can be separated by a continuous linear functional.*

Proof This follows immediately from the preceding lemmas and the Basic Separation Theorem. □

Example 3 (Tukey): The non-empty interior hypothesis is necessary for linear topological spaces. We will produce two disjoint closed, convex sets in l_2, A and B, such that $A - B$ is dense in the whole space. This implies $A - B$ connot be separated from 0 and thus A and B cannot be separated. For $x \in l_2$, we write $x = (x_n)$.

Let $A = \{x \in l_2 : x_1 \ \geq \mid n^2(xn - \frac{1}{n}) \mid \forall n > 1\}$ and let $B = \{x \in l_2 : x_n = 0 \ \ \forall n > 1\}$. A is easily seen to be closed and convex. To see that A and B are disjoint, assume some point y were in A and B. Then $y1 \ \geq \mid n^2(0 - \frac{1}{n}) \mid = n$ for $n \ \geq \ 2$ and thus y_1 is infinite, which is impossible.

To show that $A - B$ is dense, let z be an arbitrary point in l_2 and let $\epsilon > 0$ be given. Pick N so large that $\sum_{i=N} n^{-2}$ and $\sum_{i=N} z_i^2$ are each less than $\frac{\epsilon^2}{4}$. Pick x_1 so that $x_1 \ \geq \mid n^2(z_n - \frac{1}{n}) \mid$ for $2 \ \leq \ N \ < \ n$. Let $x_n = z_n$ for $2 \ \leq \ n \ < \ N$ and let $x_n = \frac{1}{n}$ for $N \ \leq \ n$. Set $y_1 = z_1 + x_1$ and $y_n = 0$ for $n > 1$. Obviously $x \in A$ and $y \in B$ and so $x - y \in A - B$. By using the triangle inequality its clear that $\parallel z - (x - y) \parallel < \ \epsilon$. Since z and ϵ were arbitrary, $A - B$ is dense.

Example 3': The above example can easily be modified to show that if X is an infinite dimensional Banach space then X contains a line B and a closed convex set A which cannot be separated. Indeed, let $\{e_n\}$ be a normalized basic sequence in X, let E be the closed linear span of $\{e_n\}$, and set $B = \langle e_1 \rangle$. Set

$$A = \{x \in E : x = \sum x_n e_n \Rightarrow x_1 \ \geq \ n^3 \mid x_n - n^{-2} \mid \forall n \ \geq \ 2\}.$$

As above, it can easily be shown that A and B are disjoint convex subsets of E with $A - B$ dense in E. Thus A and B cannot be separated by an element of E^* and hence cannot be separated by an element of X^*.

7 Locally Convex Linear Topological Spaces

Proposition 6 *Let X be a locally convex linear topological space. Disjoint convex sets B and A can be strictly separated by a continuous linear functional iff $\overline{A-B} \not\ni 0$.*

Proof If $0 \notin \overline{A-B}$ then here exists a convex open set U about 0 that does not intersect $\overline{A-B}$. By Corollary 1, there exists a continuous linear functional f separating U and $\overline{A-B}$. There is a constant δ such that $f \geq \delta$ on $\overline{A-B}$ and $f \leq \delta$ on U. Since f is non-trivial, there exists an x such that $f(x)=1$ and $f(\alpha x)=\alpha$. Since 0 is an internal point of U, $\alpha x \in U$ for sufficiently small α, say $|\alpha| < \epsilon < \delta$ for some $\epsilon > 0$. Thus $f(0) = 0 < \delta \leq \inf\{f(x) : x \in \overline{A-B}\}$. The supremum of f on B must be at least ϵ less than the infimum of f on A.

Conversely, if $0 \in \overline{A-B}$, $\{0\}$ and $A-B$ cannot be strictly separated by a continuous linear functional since 0 is a limit point of $A-B$. □

Corollary 2 *Let A and B be disjoint convex sets in a locally convex linear topological space. If A is compact and B is closed, A and B can be strictly separated.*

Proof In a topological group, the algebraic difference of a compact set and a closed set is closed. Since A and B are disjoint, $A-B$ does not contain the point 0. □

Example 4: Let X be a non-reflexive Banach space. There exist disjoint closed convex sets in X whose difference is not closed. Indeed, there exists a continuous linear functional f with unit norm that does not attain its norm on B, the closed unit ball in X (see [J]). Let $M = f^{-1}(1)$. M and B are closed sets. $0 \notin M - B$ but there exists a sequence of points in $M-B$ converging to 0, hence $M-B$ is not closed. By intersecting the hyperplane M with $2B$, this counterexample can be strengthened to yield disjoint closed bounded convex sets whose difference is not closed. These sets can be separated but cannot be strictly separated. This cannot happen in reflexive Banach spaces (see Theorem 5 below). In Theorem 6 we will show that even separation cannot always be achieved in a non-reflexive Banach space.

Corollary 3 *If A is a convex subset of a locally convex linear topological space X , and $p \notin \overline{A}$, then p and A may be strictly separated.*

Corollary 4 *If two locally convex topologies for X have the same continuous linear functionals, they have the same closed convex sets.*

Proof Let τ_1 and τ_2 be two locally convex topologies on X . Let K be a convex set closed in τ_1 and let $p \notin K$. By corollary 4 , there exists a τ_1-continuous linear functional f such that $\sup f(x) : x \in K \leq g < f(p)$. Since f is also τ_2-continuous, $\{x : | f(x) - f(p) | < f(p) - g\}$ is a τ_2 open neighborhood of p which is disjoint from K. Thus K is closed in τ_2. □

Corollary 5 *If a convex set in a Banach space is norm closed, it is closed in the weak topology.*

Proof In any Banach space, the norm and weak topologies have the same continuous linear functionals. Also, both topologies are locally convex. □

8 Banach Spaces

Theorem 5 (Tukey) *If X is a reflexive Banach space and A and B are closed disjoint convex subsets with A bounded, then A and B can be strictly separated.*

Proof By Corollary 5, a norm closed convex set is closed in the weak topology. In a reflexive space, the unit sphere is weakly compact. This implies a closed bounded convex set is weakly compact. Since the weak topology is locally convex, and A is compact in this topology, by Corollary 2, A and B may be strictly separated by a weakly continuous (hence norm continuous) functional. □

The requirement that at least one set be bounded is necessary by Example 3 above.

Theorem 6 (Klee) *Let E be a non-reflexive Banach space. Then E contains two disjoint closed bounded convex sets which cannot be separated by a closed hyperplane.*

Proof The proof will be divided into ten steps. By the Eberlein-Smulian theorem E contains a separable non-reflexive subspace. Thus without loss of generality, we may assume that E is separable.

1. *E contains a closed subspace X of infinite co-dimension such that X is non-reflexive.*

By the Eberlein-Smulian theorem, E contains a sequence $\{x_n\}$ such that $\{x_n\}$ has no weakly convergent subsequence. In particular (see [P] and [R]) there exists a subsequence $\{x'_n\}$ which is basic. Let X be the closed linear span of $\{x'_{2n}\}$.

2. Since E is separable, S_E, the unit sphere of E, contains a countable dense subset $\{e_n\}$. Define

$$C = \{0, \pm e_1, \pm\frac{1}{2}e_2, \pm\frac{1}{3}e_3, \ldots\}.$$

C is a convergent sequence with limit point and thus is compact. Let K be the closed convex hull of C. By Mazur's theorem, K is compact.

3. *Let $C = B_X$ and define $A = co(C, K)$. Claim: A is closed.*

Let $\{a_n\}$ be a sequence of points in A,

$$a_n = \lambda_n c_n + (1 - \lambda n)k_n, \ \ 0 \ \le \ \lambda_n \ \le \ 1,$$

with $c_n \in C$ and $k_n \in K$ for all n, and suppose a_n converges to a. We will show $a \in A$. By passing to a subsequence we may assume λ_n converges to λ and k_n converges to $k \in K$ by the compactness of K. If $\lambda = 0$, $a = k \in K$ and thus $a \in A$. If $\lambda > 0$, then $\lambda \neq 0$ for large n and $c_n = \frac{1}{\lambda_n}[a_n - (1 - \lambda)k_n]$

and thus the c_n's converge to $\frac{1}{\lambda}[a-(1-\lambda)k]$. Call this last expression c. $c \in C$ since C is closed. Thus $a = \lambda c + (1-\lambda)k \in co(C,K) = A$.

4. *There does not exist a closed hyperplane H such that $0 \in H$ and that A lies on one side of H.*

Suppose H were such a hyperplane. H is the kernel of some continuous linear functional f on E, $f \neq 0$. Suppose A is contained in $\{x : f(x) \leq 0\}$. Then neither $f(e_i)$ nor $f(-e_i)$ is positive and thus $f(e_i) = 0$ for all $\{e_i\}$. Since $\{e_i\}$ is dense in S_E, the closed linear span of $\{e_i\}$ is E and thus $f = 0$ by continuity.

5. *Let $D = co(X,K)$. Then $0 \in D$ but 0 is not in the interior of D. Also, D is closed.*

By substituting X for C, the proof of part 3 shows that D is closed. Let $\epsilon > 0$ be given and let $U = B_\epsilon(0)$. Suppose U is contained in D. Choose an integer n with $n+1 > \frac{2}{\epsilon}$. Let X' be the closed linear span of $\{X, e_1, e_2, e_3, \ldots, e_n\}$. Let K' be the closed convex hull of $\{\pm\frac{1}{\epsilon_1}\}$, $n < i < \infty$. Let $D' = co(X',K')$. Clearly D is contained in D' and so U is contained in D'.

Since X has infinite co-dimension, X' is a proper closed subspace of E. Thus E contains a point e of unit norm with $d(e,X') > \frac{1}{2}$ and thus $d(\epsilon e, X') > \frac{1}{2}\epsilon$. $\epsilon e \in U$ and thus $\epsilon e = \lambda x' + (1-\lambda)k'$ for some $0 \leq \lambda \leq 1$, $x' \in X'$, $k' \in K'$.

$$\| \epsilon e - \lambda x' \| = \| (1-\lambda)k' \| \leq \| k' \| \leq \frac{1}{n+1}$$

by the definition of K'. But $\lambda x'$ is a point in X', thus $\| \epsilon e - \lambda x' \| \geq d(\epsilon e, X) > \frac{\epsilon}{2}$. This establishes a contradiction.

6. *The origin is not an internal point of D. Thus there exists an $e_0 \in E$ such that $te_0 \notin D$ $\forall t > 0$.*

If 0 were an internal point of D, then E would be the union of nD as n ranges from 1 to ∞. But nD is closed and E is a complete metric space and so for some n, nD contains an interior point by the Baire category theorem. Thus D contains a point a such that $\| z \| < \epsilon$ implies $a + z \in D$ for some $\epsilon > 0$. But D is symmetric and thus $\| z \| < \epsilon$ implies $-a + z \in D$. For $\| z \| < \epsilon$, $z = \frac{1}{2}(a+z) + \frac{1}{2}(-a+z) \in D$ by convexity. This means 0 is an interior point of D, contradicting part 5.

7. *$\frac{1}{2}C$ contains a decreasing sequence $\{C_n\}$ of non-empty closed bounded convex sets with empty intersection.*

If X were reflexive, the intersection of such sets would be non-empty by weak compactness. Since X is non-reflexive, there is a continuous function f of unit norm on X such that f does not attain its norm (see [J]). Let C'_n be the intersection of C with $\{x : f(x) \geq 1 - \frac{1}{n}\}$. If there were a point in the intersection of the C_n's, that would be a point where f attains its norm. Let $C_n = \frac{1}{2}C'_n$ for all n.

8.*For all $t > 0$, $te_0 + C$ is disjoint from A.*

Suppose $te_0 + c = \lambda c' + (1-\lambda)k$, for some $0 \leq \lambda \leq 1$, c and $c' \in C$ and $k \in K$. If $\lambda > 0$, $te_0 = \lambda[c' - \frac{1}{\lambda}c] + (1-\lambda)kN \in co(X,K)$, contradicting part 6. If $\lambda = 0$, then $te_0 = -c + k$. Let

$d_n = \frac{1}{n}(-nc) + (1 - \frac{1}{n})k \in D$. Clearly the sequence d_n converges to $-c + k \in D$ since D is closed. Thus $te_0 \in D$, again contradicting part 6.

Define B to be the closed convex hull of Q, where $Q = \{e_0 + C_1, \frac{1}{2}e_0 + C_2, \frac{1}{3}e_0 + C_3 \ldots\}$. Define B' as the union of all $te_0 + C$ with $t \in (0, 1]$.

9. *Claim: B is contained in B'. Thus by part 8, B is bounded and B is disjoint from A.*

B' is convex and $\overline{B'} = B' \cup C$. Thus it suffices to show that if y is any limit point of $co(Q)$, then $y \notin C$. Let $y_m = \sum_i \lambda_{m,i}(\frac{1}{i}e_0 + d_{m,i})$, where $\sum_i \lambda_{m,i} = 1$ for each m, $d_{m,i} \in C_i$, $\lambda_{m,i} \geq 0$, and finitely many $\lambda_{m,i}$'s are non-zero for each m. Thus $y_m = t_m e_o + d_m$ where $t_m = \sum_i \frac{1}{i}\lambda_{m,i}$ and $d_m = \sum \lambda_{m,i}d_{m,i}$. Suppose the y_m's converge to y. Since $0 \leq t_m \leq 1$, by passing to a subsequence, we may assume that t_m converges to t.

Case 1: $t > 0$. Then since $d_m \in C$ for all m, the sequence of d_m's converges to a point $d \in C$ and thus the y_m's converge to $te_0 + d \in B'$.

Case 2: $t = 0$. We will produce a point in the intersections of the C_n's, contradicting part 7. Since $t = 0$, for all fixed i, $\lambda_{m,i}$ goes to zero as m goes to 0. Let $\{\epsilon_n\}$ be a sequence of positive reals converging monotonically to 0. Fix n and choose m so large that $\alpha < \epsilon_n$ where $\alpha = \sum_{i=1} \lambda_{m,i}$ and such that $t_m < \epsilon_n$.

Let $y'_m = \alpha d_{m,n} + \lambda_{m,n+1}d_{m,n+1} + \lambda_{m,n+2}d_{m,n+2} + \lambda_{m,n+3}d_{m,n+3} + \ldots$. Each y'_m is a convex combination of points in C_n, thus $y'_m \in C_n$. Also,

$$\| y_m - y'_m \| = \| t_m e_0 + \sum_{i=1} \lambda_{m,i}d_{m,i} - (\alpha d_{m,n}) \| .$$

Hence

$$\| y_m - y'_m \| \leq \epsilon_n \| e_0 \| + 2\alpha \leq \epsilon_n(2 + \| e_0 \|).$$

Call the m thus obtained $m(n)$ since the choice of m depends on n. By taking $m(1) < m(2) < m(3) \ldots$, the sequence $y'_{m(n)}$ converges to y as well. Since $y'_{m(n)} \in C_n$, $d \in C_n$ for all n.

10.*No closed hyperplane separates A from B.*

Suppose there were some continuous linear functional f, $f \neq 0$, with $f(a) \leq \lambda$ for all $a \in A$ and $f(b) \geq \lambda$ for all $b \in B$. By part 4, it suffices to show that $\lambda = 0$. Since $0 \in A$, $\lambda \geq 0$. Suppose $\lambda > 0$. Now C is contained in A and $d(B, \frac{1}{2}C) = 0$ from the definition of B. Thus there exist points $c_n \in \frac{1}{2}C$ with $f(c_n)$ converging to λ. But then $2c_n \in A$ and $f(2c_n)$ converges to $2\lambda > \lambda$, a contradiction.

□

9 Euclidian Spaces

Lemma 4 *Convex subsets of E^n have non-empty relative interior.*

Proof Let K be a convex set in E^n. By translation, we may suppose that $0 \in K$. Let S be the span of K and suppose S has dimension m. Then K contains m linearly independent vectors. Since K is convex, it contains the convex hull of these vectors, and their convex hull is an m-simplex. But an m-simplex has non-empty interior. □

This is the characteristic of Euclidean space that allows the deletion of the non-empty interior hypothesis. Although the non-empty interior hypothesis is not explicit in the locally convex case, the proof uses the fact that there exists a convex set about the origin with non-empty interior not intersecting the set under consideration.

Example 5: More general linear topological spaces fail to have this property. For example, let K be the Hilbert cube in l_2,*i.e.* the set of points in l_2 whose n^{th} coordinate has absolute value 2^{n-1}, expressed in terms of the standard basis $\{e_1, e_2, e_3 \ldots\}$. Since $0 \in K$, the affine span of K is the linear span of K. Let $y \in K$ and let $U = B_\epsilon(y) \cap \langle K\rangle$. We will show that U contains a point not in K. There exists an integer n such that $\frac{1}{n} < \delta < \epsilon$. Let $z = y + sgn(y_n)\delta e_n$. Since $\frac{1}{n}e_n \in K$, $z \in \langle K\rangle$. Also, $z \in U$ because $\| y - z \| = \delta < \epsilon$. But $z \notin K$ because its n^{th} coordinate has absolute value greater than $\frac{1}{n}$.

Lemma 5 *The interior of a convex set in E^n is a convex set.*

Proof Let K be a convex set in E^n with interior points x and y. Pick $\epsilon > 0$ so that $B_\epsilon(x) \cup B_\epsilon(y) \subseteq K$. Then for $0 \leq \lambda \leq 1$, $\lambda B_\epsilon(x) + (1-\lambda)B_\epsilon(y) = B_\epsilon(\lambda x + (1-\lambda)y) \subseteq K$. Therefore $\lambda x + (1-\lambda)y$ is in the interior of K. □

Lemma 6 *The relative interior of a convex set $K \subseteq E^n$ may be separated from any point disjoint from it.*

Proof Let C be the relative interior of K. By translation we may assume without loss of generality that C contains the origin. Select $p \notin C$. If $p \notin \langle C\rangle$ then C is in the kernel of some linear functional f for which $f(p) = 1$. Assume $p \in \langle C\rangle$. C is a convex set with non-empty interior in $\langle C\rangle$ and the point p is a convex set disjoint from it. By Corollary 1, C and p can be separated by a linear functional f defined on $\langle C\rangle$. Extend f linearly to E^n. □

Theorem 7 *In a Euclidean space, all disjoint convex sets may be separated.*

Proof Let A and B be disjoint convex subsets of E^n. $A - B$ is a convex set and does not contain 0 since A and B are disjoint. By Lemma 6, the relative interior of $A - B$ may be separated from 0. By continuity $A - B$ and 0 may also be separated. Finally, by Lemma 1, A and B can be separated. □

10 Conclusions

The Basic Separation Theorem of this paper applies to all real linear spaces: two disjoint convex sets, one of which contains an internal point, can be separated. The example of Goffman and Pedrick, Example 1, shows that it has the weakest hypothesis possible for general linear spaces. Only when we assume that our linear space is finite dimensional is it possible to completely drop the internal point requirement without adding additional hypothesis.

In a linear topological space, interior points are internal points. Also, linear functionals that separate convex sets turn out to be continuous, provided one of the convex sets has non-empty interior. Thus any two disjoint convex setsmay be separated by a closed hyperplane provided one of the sets has non-empty interior.

In locally convex linear topological spaces, the assumption that either convex set have an interior point is dropped. In this setting, two disjoint convex sets may be separated by a closed hyperplane if 0 is not a limit point of their algebraic difference. However, in the proof we introduce an open convex set containing 0 (such a set exists by local convexity) and use the topological version of the Basic Separation Theorem (Corollary 1).

If two disjoint convex sets are not closed, then in general there is no hope of strictly separating them. If they share a limit point, continuity keeps such sets from being strictly separated by a continuous functional. Also, if one of the sets has an interior point, any linear functional separating them must be continuous. Being closed does not insure strict separation either, as an example in the plane illustrates (Example 2). However, being closed and bounded in a reflexive Banach space does assure strict separation. Being closed and bounded in a non-reflexive space does not insure separation.

11 References

[BP] Barbu, V. and Precupanu, Th. “Convexity and Optimization in Banach Spaces”, D. Reidel Publishing Co., Dordrecht, 1986.

[DS] Dunford, N. and Schwartz, J. “Linear Operators, part I”, Wiley and Sons, New York, 1976.

[GP] Goffman, C. and Pedrick, G. “First Course in Functional Analysis”, Chelsea Publishing Co., New York, 1983.

[J] James, R. C. *Reflexivity and the supremum of linear functionals.* Ann. of Math., 66, 159-169 (1957).

[K] Klee, V. L., Jr. *Convex sets in linear spaces, II.* Duke Math. J. 18, 875-883 (1951).

[P] Pelczynski, A. *A note on the paper of I. Singer “Basic sequences and reflexivity of Banach spaces”*, Studia Math. 21, 371-374 (1962).

[R] Rosenthal, H. *A characterization of Banach spaces containing l_1,* Proc. Nat. Acad. Sci. (U. S. A.) 71, 2411-2413 (1974).

[RR] Robertson, A and Robertson, W. “Topological Vector Spaces”, Cambridge Press, Cambridge, 1964

[T] Tukey, J. W.*Some notes on the separation of convex sets.* Portugaliae Math. 3, 95-102 (1942).