

# Theory of Interest

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Vocabulary

**Interest periods** or **conversion periods** are times when interest is applied, such as monthly, quarterly, etc.

The **periodic rate** is the annual rate divided by the number of interest periods per year.

The **nominal rate** or **annual percentage rate** is the annual percentage of interest not taking compounding into effect.

The **effective rate** is the rate of interest paid in one year, which takes compounding into account.

For example, a nominal interest rate of 12%, compounded monthly has 12 interest periods, a periodic rate of 1%, and an effective rate of 12.68%.

The **future value** of an amount is what that amount will be worth at a given point in the future given a certain rate of interest.

The **present value** of an amount is the quantity of money that would have to be invested at a prescribed interest rate to be worth that amount in the future.

For example, with the conditions of the previous example, the future value of \$100 a year later is \$112.68 and the present value of a \$112.68 payment one year in the future is \$100.

An **annuity** is a regular sequence of payments made at the end of a period. An **annuity due** is the same, except payments are made at the beginning of the period.

The **net present value** or **NPV** of an investment is the sum of the future values of all payments minus the initial investment.

## Formulas

The effective rate of a nominal interest rate  $i$  compounded  $m$  times in a year is

$$(1 + i/m)^m - 1.$$

Let  $r$  be the periodic rate and  $n$  be the number of periods. The present value of an amount  $P$  is

$$P = F(1 + r)^{-n}.$$

The future value of an amount  $P$  is

$$F = P(1 + r)^n.$$

The “Rule of 72” says that an amount invested at an annual rate of  $i\%$  will double in approximately  $72/i$  years.

The present value of an annuity of  $n$  payments of size  $R$  is

$$PVA = R \frac{1 - (1 + r)^{-n}}{r}$$

and the future value is

$$FVA = R \frac{(1 + r)^n - 1}{r}.$$

The present value of an annuity due of  $n$  payments of size  $R$  is

$$R \left( 1 + \frac{1 - (1 + r)^{-(n-1)}}{r} \right),$$

an annuity of  $n - 1$  payments plus the first payment. The future value is

$$R \left( \frac{(1 + r)^n - 1}{r} - 1 \right),$$

an annuity of  $n + 1$  payments minus the last payment.

The effective rate of a nominal rate  $r$  compounded continuously is

$$e^r - 1.$$

The present value of an amount  $P$  compounded continuously at a nominal rate  $r$  is

$$Pe^{rt}$$

at time  $t$ . The future value of an amount  $P$  similarly compounded is

$$Pe^{-rt}.$$

Suppose money is being continuously added to an annuity at an instantaneous rate  $f(t)$  over a time interval  $[0, T]$ . The present value of this annuity is

$$\int_0^T f(t)e^{-rt} dt.$$

The future value is

$$\int_0^T f(t)e^{r(T-t)} dt.$$

## Notation

Some books use the notation

$$a_{\bar{n}|r} \equiv \frac{1 - (1 + r)^{-n}}{r},$$

the present value of an annuity  $n$  payments of size 1 at a periodic rate  $r$ .

The same books use

$$s_{\bar{n}|r} \equiv \frac{(1 + r)^n - 1}{r},$$

the future value of an annuity  $n$  payments of size 1 at a periodic rate  $r$ .

## Derivations

Effective rate: If a nominal interest rate  $i$  is compounded  $m$  times in a year, a period rate of  $i/m$  is applied  $m$  times. Hence an amount  $P$  grows to  $P(1 + i/m)^m$ , an increase of  $((1 + i/m)^m - 1)P$ .

Future value: Increase  $P$  by a rate  $r$  over  $n$  periods and you have  $P(1+r)^n$  at the end. Solve  $(1+r)^n P = F$  for  $P$  to get present value.

The present value of an annuity is the sum of the present value of its payments, which is a geometric series:

$$\sum_{j=1}^n R(1+r)^{-j} = R \frac{1 - (1+r)^{-n}}{r}.$$

Future values and annuities due are handled similarly.

Continuous compounding: By taking the limit as the number of interest periods becomes infinite, we have

$$\lim_{m \rightarrow \infty} P \left(1 + \frac{r}{m}\right)^{mt} = Pe^{rt}.$$

To find out how long it takes for an amount  $P$  to double when compounded continuously, we need to solve

$$Pe^{rt} = 2P$$

for  $t$ , and we get  $t = \ln 2/r = 0.69314/r$ . Thus if we round 0.69314 to 72%, we get the “Rule of 72.” The derivation shows the limitations of the rule. First, it only applies to continuous compounding, but for moderate interest rates, the effective rate for continuous compounding is not too different from, say, quarterly compounding. Just for the fun of it, we note that from the Taylor series for  $e^r$  we see that the difference between continuous and quarterly compounding is

$$e^r - \left(1 + \frac{r}{4}\right)^4 = \frac{r^2}{8} + \mathcal{O}(r^3).$$

Thus even for interest rates around 20%, the difference is in the neighborhood of half a percent.

Second, it should be the “Rule of  $100 \ln 2$ ,” or “Rule of 69.314” if we’re willing to truncate a decimal. But 72 is a nice number to work with since it is divisible by a lot of integers. Substituting 72 for  $100 \ln 2$  makes our estimates about 4% too pessimistic.

Continuous annuities: Note that the present value of a payment  $f(t)$  is  $f(t)e^{-rt}$ ; integration adds these up. For future value, note that each payment  $f(t)$  earns interest for  $T - t$  years and so has a future value  $f(t)e^{r(T-t)}$ . Note that

$$\int_0^T f(t)e^{r(T-t)} dt = e^{rT} \int_0^T f(t)e^{-rt} dt,$$

which says that we could find the future value of annuity by first finding its present value and then finding the future value of a lump sum equal to that amount. On a more esoteric note, we observe that as  $T$  becomes large, the present value of  $f(t)$  approaches the Laplace transform of  $f(t)$ .