Richard Stanley's Twelvefold Way

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Many combinatorial problems can be framed as counting the number of ways to allocate balls to urns, subject to various conditions. Richard Stanley invented the "twelvefold way" to organize these results into a table with twelve entries. See his book *Enumerative Combinatorics, Volume 1.*

Let b represent the number of balls available and u the number of urns. The following table gives the number of ways to partition the balls among the urns according to the various states of labeled or unlabeled and subject to certain restrictions. The column headed "≤ 1" corresponds to requiring that there be no more than one ball in each urn. Similarly, the column headed "≥ 1" corresponds to requiring at least one ball in each urn.

Balls	Urns	unrestricted	≤ 1	≥ 1
labeled	labeled	u^b	$(u)_b$	$u!S(b,u)$
unlabeled	labeled	$\left(\binom{u}{b}\right)$	$\binom{u}{b}$	$\left(\binom{u}{b-u}\right)$
labeled	unlabeled	$\sum_{i=1}^{u} S(b,i)$	$[b \leq u]$	$S(b,u)$
unlabeled	unlabeled	$\sum_{i=1}^{u} p_i(b)$	$[b \leq u]$	$p_u(b)$

For convenient cross referencing, we will refer to each of the cases by three symbols. The first character is an `l` or a `u` depending on whether the balls are labeled or unlabeled. The second character similarly indicates whether the urns are labeled or unlabeled. The final character is one the regular expression symbols $*$, ?, or + indicating no restrictions, at most one ball per urn, and at least one ball per urn respectively.

1 Labeled balls, labeled urns, unrestricted (`ll*`)

This is the number of b-tuples of u things. There are u choices for which urn the first ball should go in, u choices for where the second ball should go, and so forth for a total of u^b possibilities.

2 Labeled balls, labeled urns, ≤ 1 (`ll?`)

There are u choices for where to place the first ball, $u-1$ for where to place the second ball, and so forth for a total of

$$u(u-1)\cdots(u-b+1) = (u)_b$$

possibilities. $(u)_b$ is also written $u^{\underline{b}}$. This is sometimes read "u to b factors" or "the bth falling power of u."

Note that if $u = b$ then $(u)_b = u!$. Also, if $b > u$ then $(u)_b = 0$ because there are too many balls for each urn to hold only one.

3 Unlabeled balls, labeled urns, unrestricted (`ul*`)

The number of ways to choose k things from a set of size n is $\binom{n}{k}$. This is selection without replacement. By analogy, Stanley defines $\left(\binom{n}{k}\right)$ to be the number of ways to choose k items from a set of size n *with replacement*. The number of ways to distribute b unlabeled balls into u urns is $\left(\binom{u}{b}\right)$ because for each ball one can choose which urn to put it in. Since an urn can be selected several times, this is choosing with replacement. (It is important to think of choosing urns, not balls.)

Another way of viewing the analogy above is that whereas $\binom{n}{k}$ gives the number of subsets of size k from a set of size n, $\left(\!\binom{n}{k}\!\right)$ gives the number of k-element multisets drawn on a set of n elements.

It turns out that

$$\left(\!\!\binom{n}{k}\!\!\right) = \binom{n+k-1}{k}$$

and so the number of ways to distribute b unlabeled balls into u urns is $\binom{u+b-1}{b}$. This can be proven via Feller's stars-and-bars argument as follows.

Imagine the u urns as the spaces between $u+1$ vertical bars. Represent the balls as stars between the bars. For example, |*||***| gives an illustration of one way to assign four balls to three cells. There must be a bar in the first and last position, but otherwise there are as many ways to assign b balls to u urns as there are ways to arrange the stars and bars. There are $u-1$ bars that we are free to move and b stars, for a total of $u+b-1$ symbols. Among the $u+b-1$ positions for these symbols, we choose b in which to put stars and fill the rest with bars. Thus there are $\binom{u+b-1}{b}$ possibilities.

In some sense Stanley's notation is unnecessary since it easily reduces to binomial coefficients. However, some equations are cleaner and more memorable using his notation.

$\left(\!\binom{n}{k}\!\right)$ is also the number of solutions to the equation

$$x_1 + x_2 + \cdots x_k = n + k$$

in positive integers.

4 Unlabeled balls, labeled urns, ≤ 1 (ul?)

If there can be no more than one ball in each urn, the process reduces to determining which b of the u urns will get a ball, and there are $\binom{u}{b}$ ways to choose b urns out of the total of u urns.

5 Unlabeled balls, labeled urns, ≥ 1 (ul+)

Stanley gives the solution to this problem as $\left(\binom{u}{b-u}\right)$ which reduces to $\binom{b-1}{u-1}$.

Stanley's derivation is as follows. Put one ball in each urn. Now there are $b-u$ balls that can be distributed without restriction and so by problem (ul*), this can be done in $\left(\binom{u}{b-u}\right)$ ways.

Feller derives the same result directly as follows. Imagine the u urns as the spaces between $u+1$ bars. We first think of the b stars lined up and decide where to place the bars. We must place one bar before the first star and one after the last star. The remaining $u-1$ bars must be distributed among the $b-1$ spaces between stars. Thus there are $\binom{b-1}{u-1}$ possibilities.

6 Labeled balls, unlabeled urns, ≥ 1 (lu+)

The notation $S(n,k)$ denotes Stirling numbers of the second kind. Knuth uses a notation – I have not found out yet how to reproduce it in LaTeX– that looks like $\binom{n}{k}$ with the parentheses replaced with curly braces $\{\}$. Knuth pronounces his symbol "n subset k" and calls the Stirling numbers of the second kind "subset numbers." His notation and pronunciation are more mnemonic than their traditional counterparts. $S(n,k)$ is defined to be the number of ways to partition n objects into k non-empty, unordered sets. For example, $S(4,2)=7$ because there are seven ways to partition $\{1,2,3,4\}$ into two sets:

- $\{1\} \cup \{2, 3, 4\}$
- $\{2\} \cup \{1, 3, 4\}$
- $\{3\} \cup \{1, 2, 4\}$
- $\{4\} \cup \{1, 2, 3\}$
- $\{1, 2\} \cup \{3, 4\}$
- $\{1, 3\} \cup \{2, 4\}$
- $\{1, 4\} \cup \{2, 3\}$

Partitioning into u subsets corresponds to putting balls in u urns. Requiring the subsets to be non-empty corresponds to requiring each urn to have at least one ball. Unordered subsets correspond to unlabeled urns. Therefore there are $S(b, u)$ possibilities.

There is no convenient formula for Stirling numbers, but they may be computed via the recurrence relationship

$$S(n, k) = k\, S(n-1, k) + S(n-1, k-1).$$

7 Labeled balls, labeled urns, ≥ 1 (`ll+`)

See the development of (`lu+`). Assume first that the urns are not labeled. Then there are $S(b, u)$ possibilities. But since the urns can be permuted $u!$ ways among themselves, labeling the urns multiplies the number of possibilities by $u!$.

8 Labeled balls, unlabeled urns, unrestricted (`lu*`)

By (`lu+`) there are $S(b, u)$ ways to distribute the balls into u subsets with at least one ball in each subset. If we allow the number of subsets to vary from 1 to u, then the total number of possibilities is the sum of over each size:

$$S(b, 1) + S(b, 2) + \cdots + S(b, u).$$

9 Labeled balls, unlabeled urns, ≤ 1 (`lu?`)

If there are more balls than urns, there is no way to partition the balls into the urns with no more than one ball per urn. If the number of balls is less than or equal to the number of urns, there is one solution: put one ball in each urn. Since the urns are unlabeled, one cannot distinguish any difference between different ways of doing this. Therefore the number of ways to distribute b (unlabeled) balls into u unlabeled urns with no more than one ball in each urn is $[b \leq u]$, *i.e.* 1 if $b \leq u$ and 0 otherwise.

10 Unlabeled balls, unlabeled urns, ≤ 1 (uu?)

The argument in (`lu?`) holds here as well: if the urns are not labeled, it doesn't matter that the balls are labeled. If ball 1 is in one unlabeled urn and ball 2 is in another unlabeled urn, how could I tell if I switched the two balls?

11 Unlabeled balls, unlabeled urns, ≥ 1 (uu+)

$p_k(n)$ is defined as the number of partitions of n into k parts, *i.e.* the number of distinct ways to write n as the sum of k positive integers. The order of the summands does not matter; equivalently one can require that the summands be arranged in non-decreasing order.

Partitioning b non-distinct balls into u non-distinct urns corresponds to partitioning b into u summands. Requiring at least one ball in each urn corresponds to requiring the summands to be positive. Therefore b unlabeled balls can be distributed into u unlabeled urns in $p_u(b)$ ways.

There is no convenient formula for the numbers $p_k(n)$, but they can be computed via the recurrence relation

$$p_k(n) = p_{k-1}(n-1) + p_k(n-k).$$

12 Unlabeled balls, unlabeled urns, unrestricted (uu∗)

Without the requirement of at least one ball per urn, the number of non-empty urns can be any number between 1 and u inclusive. Thus, using (uu+), the number of ways to distribute b unlabeled balls into u unlabeled urns is

$$p_1(b) + p_2(b) + \cdots + p_u(b).$$