The Method of Undeterminded Coefficients

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Undeterminded coefficients may be the most complicated thing we've done this semester. Yet it can be summarized in one paragraph.

Suppose we have an Nth order linear equation with constant coefficients

$$\sum_{k=0}^{N} a_k y^{(k)} = f(x).$$

Suppose

$$f(x) = P(x)e^{\alpha x} \sin \beta x$$

or

$$f(x) = P(x)e^{\alpha x} \cos \beta x$$

where $P(x)$ is an nth degree polynomial. Then our differential equation has a particular solution of the form

$$y_p = x^s e^{\alpha x} \left(A(x) \sin \beta x + B(x) \cos \beta x \right).$$

Here s is the smallest non-negative integer such that no term in y_p is a solution to the homogeneous problem,

$$A(x) = A_0 x^n + A_1 x^{n-1} + \ldots + A_n,$$

and

$$B(x) = B_0 x^n + B_1 x^{n-1} + \ldots + B_n.$$

That's it! Admittedly, it's a complicated paragraph, but at least one can gather it all into one place. A few comments will show how everything we've done in regard to undetermined coefficients is included.

If there are no exponential terms, set $\alpha = 0$. If there are no polynomial terms, let $n = 0$; the constant "1" is a zeroth degree polynomial. If there are no trig functions, set $\beta = 0$. If no homogeneous solutions appear on the right side, $s = 0$.

EXAMPLE: In the equation $y'' + y = x^2$, we have $\alpha = 0$, $\beta = 0$. Thus $y_p = B_0 x^2 + B_1 x + B_2$.

EXAMPLE: In the equation $y'' + y = \cos 3x$, we have $\alpha = 0$, $\beta = 3$, $P(x) = 1$ and $n = 0$. Thus $y_p = A_0 \sin 3x + B_0 \cos 3x$.

EXAMPLE: In the equation $y'' + y = x \cos x$, we have $\alpha = 0$, $\beta = 1$, and $P(x) = x$. Thus $y_p = x\left((A_0 x + A_1)\sin x + (B_0 x + B_1)\cos x\right)$. Note that since $A_1 \sin x$ and $B_1 \cos x$ are homogeneous solutions we must have $s = 1$. Also, since $P(x)$ is a first degree polynomial, $\sin x$ and $\cos x$ must have general first degree terms in front of them.

EXAMPLE: In the equation $y^{(6)} + 2y^{(4)} + y'' = xe^{-x}$, we have $\alpha = -1$, $P(x) = x$, and $\beta = 0$. Thus $y_p = x^2(B_0 x + B_1)e^{-x}$. Note that $s = 2$ because xe^{-x} is a homogeneous solution.

Some common mistakes to avoid:

1. Even if $f(x) = x^n$, it is necessary to solve for $A_0 \ldots A_n$, not just A_0.
2. If sine *or* cosine appear on the right side, sine *and* cosine must appear in y_p.
3. The value of s must be chosen so that *no term* in y_p is a homogeneous solution.