Abstract PDE Operators

John D. Cook

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Continuity Conditions

continuous	$u_n \to u \Rightarrow Au_n \to Au$
completely continuous	$u_n \rightharpoonup u \Rightarrow Au_n \to Au$
weakly continuous	$u_n \rightharpoonup u \Rightarrow Au_n \rightharpoonup Au$
demi-continuous	$u_n \to u \Rightarrow Au_n \rightharpoonup Au$
compact	S bounded $\Rightarrow \overline{A(S)}$ compact
bounded	S bounded $\Rightarrow A(S)$ bounded
locally bounded	$u_n \to u \Rightarrow \{Au_n\}$ bounded
hemi-continuous	For all u,v, $t \mapsto A(u+tv)v$ is continuous.
type-M	$u_n \rightharpoonup u$, $Au_n \rightharpoonup f$, and $\limsup \langle Au_n, u_n \rangle \leq \langle f, u \rangle \Rightarrow Au = f$.

A linear operator on a reflexive Banach space is continuous iff it is weakly continuous and compact iff it is completely continuous.

Monotonicity Conditions

monotone	$\langle Au - Av, u - v \rangle \geq 0$
strongly monotone	$\langle Au - Av, u - v \rangle \geq c \parallel u - v \parallel^2$
maximal monotone	$\langle Av - f, v - u \rangle \geq 0 \;\; \forall v \Rightarrow Au = f$
pseudomonotone	$u_n \rightharpoonup u$ and $\limsup \langle Au_n, u_n - u \rangle \leq 0 \Rightarrow \langle Au, u - v \rangle \leq \langle Au_n, u_n - u \rangle$

Continuity Theorems

A monotone hemi-continuous operator is maximal monotone and pseudomonotone.
Pseudomonotone or maximal monotone imply type-M.
A maximal monotone operator is locally bounded.
A locally bounded type-M operator is demi-continuous.

If A is type-M and B is (completely continuous) or (monotone and weakly continuous) then $A + B$ is type-M.

Embedding Theorems

$n < mp$ and $1 \leq q \leq \infty \Rightarrow W^{m,p}(\Omega) \overset{cpt}{\hookrightarrow} C_B(\Omega) \hookrightarrow L^q(\Omega)$

$n = mp$ and $1 \leq q < \infty \Rightarrow W^{m,p}(\Omega) \overset{cpt}{\hookrightarrow} L^q(\Omega)$

$n > mp$ and $q = \frac{np}{n-mp} \Rightarrow W^{m,p}(\Omega) \overset{cpt}{\hookrightarrow} L^q(\Omega)$

$n > mp$ and $1 \leq q < \frac{np}{n-mp} \Rightarrow W^{m,p}(\Omega) \overset{cpt}{\hookrightarrow} L^q(\Omega)$