

Asymptotic results for Normal-Cauchy model

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Abstract

This report proves asymptotic results for the posterior mean when sampling from a normal distribution with a Cauchy prior on the location parameter.

1 Problem statement

Define

$$\phi(\theta; \sigma) = \frac{1}{\sqrt{2\pi}\sigma} \exp\left(-\frac{\theta^2}{2\sigma^2}\right)$$

and

$$c(\theta) = \frac{1}{\pi(1 + \theta^2)}.$$

Let y_i be samples from a normal(θ , σ^2) distribution where θ has a Cauchy(0, 1) prior. Let \bar{y} be the sample mean of the y_i values. The posterior mean of θ is given by

$$\frac{\int_{-\infty}^{\infty} \theta \phi(\bar{y} - \theta; \sigma/\sqrt{n}) c(\theta) d\theta}{\int_{-\infty}^{\infty} \phi(\bar{y} - \theta; \sigma/\sqrt{n}) c(\theta) d\theta}.$$

We will establish asymptotic results for the posterior mean of θ as $\bar{y} \rightarrow \infty$ and as $n \rightarrow \infty$.

2 Preliminaries

2.1 Fourier transforms

We will use the following results from Fourier analysis to evaluate the integrals below. First, define the Fourier transform of a function $f \in L^1(\mathbb{R})$ as

$$\mathcal{F}(f)(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(\theta) \exp(ix\theta) d\theta.$$

The inverse Fourier transform is then given by

$$\mathcal{F}(f)(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(\theta) \exp(-ix\theta) d\theta.$$

Define the convolution of two functions f and g by

$$(f \star g)(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(\theta) g(x - \theta) d\theta.$$

Then

$$f \star g = g \star f$$

and

$$\mathcal{F}(f \star g) = \mathcal{F}(f)\mathcal{F}(g).$$

2.2 Integrating the exponential of a quadratic

We will also use the following result.

Claim 1. For $a > 0$ and $b \in \mathbb{C}$,

$$\int_0^{\infty} \exp(-ax^2 + bx) dx = \frac{\sqrt{\pi} \exp\left(\frac{b^2}{4a}\right)}{2\sqrt{a}} \operatorname{erfc}\left(-\frac{b}{2\sqrt{a}}\right). \quad (1)$$

Proof. Use the factorization

$$ax^2 - bx = \left(\sqrt{a}x - \frac{b}{2\sqrt{a}} \right)^2 - \frac{b^2}{2a}$$

to come up with the change of variables

$$w = \sqrt{a}x - \frac{b}{2\sqrt{a}}.$$

Now

$$\begin{aligned} \int_0^\infty \exp(-ax^2 + bx) dx &= \frac{1}{\sqrt{a}} \int_{-b/2\sqrt{a}}^\infty \exp(-w^2) \exp\left(\frac{b^2}{2a}\right) dw \\ &= \frac{\exp\left(\frac{b^2}{2a}\right)}{\sqrt{a}} \int_{-b/2\sqrt{a}}^\infty \exp(-w^2) dw \\ &= \frac{\sqrt{\pi} \exp\left(\frac{b^2}{2a}\right)}{2\sqrt{a}} \frac{2}{\sqrt{\pi}} \int_{-b/2\sqrt{a}}^\infty \exp(-w^2) dw \\ &= \frac{\sqrt{\pi} \exp\left(\frac{b^2}{4a}\right)}{2\sqrt{a}} \operatorname{erfc}\left(-\frac{b}{2\sqrt{a}}\right). \end{aligned}$$

□

3 Integration results

Claim 2.

$$\int_{-\infty}^\infty \phi(y - \theta; \sigma) c(\theta) d\theta = \frac{1}{2\sqrt{2\pi}\sigma} \left\{ h\left(\frac{1+iy}{\sqrt{2}\sigma}\right) + h\left(\frac{1-iy}{\sqrt{2}\sigma}\right) \right\} \quad (2)$$

where

$$h(z) = \exp(z^2) \operatorname{erfc}(z).$$

Proof. The integral in equation (2) equals $\sqrt{2\pi}(\phi \star c)(y)$ and so its Fourier transform is $\mathcal{F}(\phi) \mathcal{F}(c)$. These Fourier transforms are well known:

$$\begin{aligned} \mathcal{F}(\phi)(x) &= \frac{1}{\sqrt{2\pi}} \exp(-\sigma^2 x^2 / 2) \\ \mathcal{F}(c)(x) &= \frac{1}{\sqrt{2\pi}} \exp(-|x|). \end{aligned}$$

Applying the inverse Fourier transform we have the following.

$$\begin{aligned}
\int_{-\infty}^{\infty} \phi(y - \theta; \sigma) c(\theta) d\theta &= \sqrt{2\pi}(\phi \star c)(y) \\
&= \mathcal{F}^{-1}(\mathcal{F}(\sqrt{2\pi}(\phi \star c)(y))) \\
&= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} \exp(-\sigma^2 x^2 - |x| - ixy) dx \\
&= \frac{1}{2\pi} \int_0^{\infty} \exp(-\sigma^2 x^2/2 - x - ixy) dx \\
&\quad + \frac{1}{2\pi} \int_{-\infty}^0 \exp(-\sigma^2 x^2/2 + x - ixy) dx.
\end{aligned}$$

The final two integrals can be evaluated in terms of the complementary error function erfc using Claim 1. \square

Claim 3.

$$\int_{-\infty}^{\infty} \theta \phi(y - \theta; \sigma) c(\theta) d\theta = \frac{i}{2\sqrt{2\pi}\sigma} \left\{ h\left(\frac{1+iy}{\sqrt{2}\sigma}\right) - h\left(\frac{1-iy}{\sqrt{2}\sigma}\right) \right\} \quad (3)$$

where as before

$$h(z) = \exp(z^2) \operatorname{erfc}(z).$$

Proof. As before we take the Fourier transform of the integral and then transform back. Define the function

$$\eta(\theta) = \frac{\theta}{\pi(1+\theta^2)}.$$

Here we use the fact that

$$\mathcal{F}(\eta)(x) = \frac{i}{\sqrt{2\pi}} \frac{x}{|x|} \exp(-|x|).$$

(The function η is not in $L^1(\mathbb{R})$ and so the elementary definition of the Fourier transform does not hold. But $\eta \in L^2(\mathbb{R})$ and when the Fourier transform is extended to $L^2(\mathbb{R})$, the right side of the equation above is its transform.)

$$\int_{-\infty}^{\infty} \theta \phi(y - \theta; \sigma) c(\theta) d\theta = \sqrt{2\pi} \phi \star \eta$$

$$\begin{aligned}
&= \sqrt{2\pi} \mathcal{F}^{-1} \left(\frac{1}{\sqrt{2\pi}} \exp\left(-\frac{\sigma^2 x^2}{2}\right) \frac{i}{\sqrt{2\pi}} \frac{x}{|x|} \exp(-|x|) \right) \\
&= i \int_{-\infty}^{\infty} \frac{1}{2\pi} \frac{x}{|x|} \exp(-\sigma^2 x^2/2 - |x| - ixy) dx \\
&= i \int_0^{\infty} \frac{1}{2\pi} \exp(-\sigma^2 x^2/2 - x - ixy) dx \\
&\quad - i \int_{-\infty}^0 \frac{1}{2\pi} \exp(-\sigma^2 x^2/2 + x - ixy) dx.
\end{aligned}$$

As before, the final two integrals can be evaluated using Claim 1. \square

4 Asymptotic results

Next we apply the asymptotic series

$$\operatorname{erfc}(z) = \frac{\exp(-z^2)}{\sqrt{\pi z}} \left(1 - \frac{1}{2z^2} + \dots \right)$$

to the integrals above. This series is valid for $|\arg(z)| < 3\pi/4$. Since we will only be interested in values of z with positive real part, the series is valid for our use.

For the right side of equation (2) the first term of the asymptotic series is sufficient.

Claim 4. As $y \rightarrow \infty$,

$$\int_{-\infty}^{\infty} \phi(y - \theta) c(\theta) d\theta \sim \frac{1}{\pi(1 + y^2)}.$$

Proof. Intuitively, as y becomes large, the function $c(\theta)$ becomes very flat. Multiplying by $\phi(y - \theta)$ and integrating essentially samples the function $c(\theta)$ at y .

To prove this assertion, apply the asymptotic approximation

$$\operatorname{erfc}(z) \sim \frac{\exp(-z^2)}{\sqrt{\pi z}}$$

to equation (2). With a little rearrangement, the arguments inside the exponential functions become zero and the integral reduces to

$$\frac{1}{2\sqrt{2\pi}\sigma} \left\{ \frac{\exp(0)}{\sqrt{\pi} \left(\frac{1-iy}{\sqrt{2}\sigma}\right)} + \frac{\exp(0)}{\sqrt{\pi} \left(\frac{1+iy}{\sqrt{2}\sigma}\right)} \right\} = \frac{1}{\pi(1+y^2)}.$$

□

Next we apply the asymptotic series for erfc to equation (3).

Claim 5. As $y \rightarrow \infty$,

$$\int_{-\infty}^{\infty} \theta \phi(y - \theta) c(\theta) d\theta \sim \frac{y}{\pi} \left\{ \frac{(1+y^2)^2 + \sigma^2(y^2 - 3)}{(1+y^2)^3} \right\}.$$

Proof. Here we apply the two-term asymptotic approximation

$$\operatorname{erfc}(z) \sim \frac{\exp(-z^2)}{\sqrt{\pi}z} \left(1 - \frac{1}{2z^2} \right).$$

As before, the arguments of the exponential functions become zero and the integral reduces to

$$\frac{1}{2\pi} \left\{ \frac{1}{1-iy} \left(1 - \left(\frac{\sigma}{1-iy} \right)^2 \right) - \frac{1}{1+iy} \left(1 - \left(\frac{\sigma}{1+iy} \right)^2 \right) \right\}$$

which further reduces to

$$\frac{y}{\pi} \left\{ \frac{(1+y^2)^2 + \sigma^2(y^2 - 3)}{(1+y^2)^3} \right\}.$$

□

Consider a single sample y from a normal(θ , σ^2) distribution where θ has a conjugate normal(0, τ^2) prior. It is well-known that the posterior distribution on θ has mean

$$\frac{\tau^2}{\tau^2 + \sigma^2} y.$$

Claim 6. *If we use a Cauchy(0, 1) prior rather than a normal(0, τ^2) prior on θ above, the posterior mean of θ is*

$$y - \mathcal{O}\left(\frac{1}{y}\right)$$

as $y \rightarrow \infty$.

Proof. Simply take the ratio of the results of Claim 5 and Claim 4. \square

Next consider taking multiple samples y_i from a normal(θ , σ^2) distribution. Denote the sample mean of the y_i values by \bar{y} . We examine the posterior mean of θ under normal and Cauchy priors as the number of samples n increases.

With a normal(0, τ^2) prior on θ , the posterior mean of θ after observing n samples with mean \bar{y} is

$$\frac{1}{1 + \frac{\sigma^2}{n\tau^2}} \bar{y} = \left(1 - \frac{\sigma^2}{n\tau^2} + \mathcal{O}\left(\frac{1}{n^2}\right)\right) \bar{y}.$$

Claim 7. *If θ has a Cauchy(0, 1) prior, then the posterior mean of θ after sampling n values with sample mean \bar{y} is*

$$\bar{y} + \frac{(\bar{y}^2 - 3)\bar{y} \sigma^2}{(1 + \bar{y}^2)^2 n} + \mathcal{O}\left(\frac{1}{n^2}\right).$$

Proof. Observing n values from a normal(θ , σ^2) distribution is the same as observing one value \bar{y} from a normal(θ , σ^2/n) distribution. The claim can be established analogous to Claim 5 letting $n \rightarrow \infty$ rather than $y \rightarrow \infty$. \square

The rate at which the posterior mean converges to \bar{y} depends on τ in the case of the normal prior and on \bar{y} in the case of the Cauchy prior. For any value of τ , the convergence is faster under the Cauchy prior for sufficiently large values of \bar{y} .

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