Asymptotic results for Normal-Cauchy model

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Abstract

This report proves asymptotic results for the posterior mean when sampling from a normal distribution with a Cauchy prior on the location parameter.

1 Problem statement

Define

$$\phi(\theta; \sigma) = \frac{1}{\sqrt{2\pi}\sigma} \exp\left(-\frac{\theta^2}{2\sigma^2}\right)$$

and

$$c(\theta) = \frac{1}{\pi(1+\theta^2)}.$$

Let y_i be samples from a normal(θ, σ^2) distribution where θ has a Cauchy(0, 1) prior. Let $\overline{y}$ be the sample mean of the y_i values. The posterior mean of θ is given by

$$\frac{\int_{-\infty}^{\infty} \theta\, \phi(\overline{y} - \theta; \sigma/\sqrt{n})\, c(\theta)\, d\theta}{\int_{-\infty}^{\infty} \phi(\overline{y} - \theta; \sigma/\sqrt{n})\, c(\theta)\, d\theta}.$$

We will establish asymptotic results for the posterior mean of θ as $\overline{y} \to \infty$ and as $n \to \infty$.

2 Preliminaries

2.1 Fourier transforms

We will use the following results from Fourier analysis to evaluate the integrals below. First, define the Fourier transform of a function $f \in L^1(\mathbb{R})$ as

$$\mathcal{F}(f)(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(\theta)\, \exp(ix\theta)\, d\theta.$$

The inverse Fourier transform is then given by

$$\mathcal{F}(f)(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(\theta)\, \exp(-ix\theta)\, d\theta.$$

Define the convolution of two functions f and g by

$$(f \star g)(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(\theta)\, g(x-\theta)\, d\theta.$$

Then

$$f \star g = g \star f$$

and

$$\mathcal{F}(f \star g) = \mathcal{F}(f)\mathcal{F}(g).$$

2.2 Integrating the exponential of a quadratic

We will also use the following result.

Claim 1. *For $a > 0$ and $b \in \mathbb{C}$,*

$$\int_0^{\infty} \exp(-ax^2 + bx)\, dx = \frac{\sqrt{\pi}\exp\left(\frac{b^2}{4a}\right)}{2\sqrt{a}} \operatorname{erfc}\left(-\frac{b}{2\sqrt{a}}\right). \tag{1}$$

Proof. Use the factorization

$$ax^2 - bx = \left(\sqrt{a}x - \frac{b}{2\sqrt{a}}\right)^2 - \frac{b^2}{2a}$$

to come up with the change of variables

$$w = \sqrt{a}x - \frac{b}{2\sqrt{a}}.$$

Now

$$\begin{aligned}
\int_0^\infty \exp(-ax^2 + bx)\, dx &= \frac{1}{\sqrt{a}} \int_{-b/2\sqrt{a}}^\infty \exp(-w^2) \exp\left(\frac{b^2}{2a}\right) dw \\
&= \frac{\exp\left(\frac{b^2}{2a}\right)}{\sqrt{a}} \int_{-b/2\sqrt{a}}^\infty \exp(-w^2)\, dw \\
&= \frac{\sqrt{\pi}\exp\left(\frac{b^2}{2a}\right)}{2\sqrt{a}} \frac{2}{\sqrt{\pi}} \int_{-b/2\sqrt{a}}^\infty \exp(-w^2)\, dw \\
&= \frac{\sqrt{\pi}\exp\left(\frac{b^2}{4a}\right)}{2\sqrt{a}} \operatorname{erfc}\left(-\frac{b}{2\sqrt{a}}\right).
\end{aligned}$$

□

3 Integration results

Claim 2.

$$\int_{-\infty}^\infty \phi(y-\theta;\sigma)\, c(\theta)\, d\theta = \frac{1}{2\sqrt{2\pi}\sigma} \left\{ h\left(\frac{1+iy}{\sqrt{2}\sigma}\right) + h\left(\frac{1-iy}{\sqrt{2}\sigma}\right) \right\} \quad (2)$$

where

$$h(z) = \exp(z^2) \operatorname{erfc}(z).$$

Proof. The integral in equation (2) equals $\sqrt{2\pi}(\phi \star c)(y)$ and so its Fourier transform is $\mathcal{F}(\phi)\,\mathcal{F}(c)$. These Fourier transforms are well known:

$$\begin{aligned}
\mathcal{F}(\phi)(x) &= \frac{1}{\sqrt{2\pi}} \exp(-\sigma^2 x^2/2) \\
\mathcal{F}(c)(x) &= \frac{1}{\sqrt{2\pi}} \exp(-|x|).
\end{aligned}$$

Applying the inverse Fourier transform we have the following.

$$
\begin{aligned}
\int_{-\infty}^{\infty} \phi(y-\theta;\sigma)\, c(\theta)\, d\theta &= \sqrt{2\pi}(\phi \star c)(y) \\
&= \mathcal{F}^{-1}(\mathcal{F}(\sqrt{2\pi}(\phi \star c)(y))) \\
&= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} \exp(-\sigma^2 x^2 - |x| - ixy)\, dx \\
&= \frac{1}{2\pi} \int_{0}^{\infty} \exp(-\sigma^2 x^2/2 - x - ixy)\, dx \\
&\quad + \frac{1}{2\pi} \int_{-\infty}^{0} \exp(-\sigma^2 x^2/2 + x - ixy)\, dx.
\end{aligned}
$$

The final two integrals can be evaluated in terms of the complementary error function erfc using Claim 1. □

Claim 3.

$$
\int_{-\infty}^{\infty} \theta\phi(y-\theta;\sigma)\, c(\theta)\, d\theta = \frac{i}{2\sqrt{2\pi}\sigma} \left\{ h\left(\frac{1+iy}{\sqrt{2}\sigma}\right) - h\left(\frac{1-iy}{\sqrt{2}\sigma}\right) \right\} \tag{3}
$$

where as before

$$
h(z) = \exp(z^2)\,\mathrm{erfc}(z).
$$

Proof. As before we take the Fourier transform of the integral and then transform back. Define the function

$$
\eta(\theta) = \frac{\theta}{\pi(1+\theta^2)}.
$$

Here we use the fact that

$$
\mathcal{F}(\eta)(x) = \frac{i}{\sqrt{2\pi}} \frac{x}{|x|} \exp(-|x|).
$$

(The function η is not in $L^1(\mathbb{R})$ and so the elementary definition of the Fourier transform does not hold. But $\eta \in L^2(\mathbb{R})$ and when the Fourier transform is extended to $L^2(\mathbb{R})$, the right side of the equation above is its transform.)

$$
\int_{-\infty}^{\infty} \theta\phi(y-\theta;\sigma)\, c(\theta)\, d\theta = \sqrt{2\pi}\phi \star \eta
$$

$$
\begin{aligned}
&= \sqrt{2\pi}\mathcal{F}^{-1}\left(\frac{1}{\sqrt{2\pi}}\exp\left(-\frac{\sigma^2 x^2}{2}\right)\frac{i}{\sqrt{2\pi}}\frac{x}{|x|}\exp(-|x|)\right) \\
&= i\int_{-\infty}^{\infty}\frac{1}{2\pi}\frac{x}{|x|}\exp(-\sigma^2x^2/2-|x|-ixy)\,dx \\
&= i\int_{0}^{\infty}\frac{1}{2\pi}\exp(-\sigma^2x^2/2-x-ixy)\,dx \\
&\quad -i\int_{-\infty}^{0}\frac{1}{2\pi}\exp(-\sigma^2x^2/2+x-ixy)\,dx.
\end{aligned}
$$

As before, the final two integrals can be evaluated using Claim 1. □

4 Asymptotic results

Next we apply the asymptotic series

$$
\operatorname{erfc}(z) = \frac{\exp(-z^2)}{\sqrt{\pi}z}\left(1-\frac{1}{2z^2}+\cdots\right)
$$

to the integrals above. This series is valid for $|\arg(z)| < 3\pi/4$. Since we will only be interested in values of z with positive real part, the series is valid for our use.

For the right side of equation (2) the first term of the asymptotic series is sufficient.

Claim 4. *As $y \to \infty$,*

$$
\int_{-\infty}^{\infty}\phi(y-\theta)\,c(\theta)\,d\theta \sim \frac{1}{\pi(1+y^2)}.
$$

Proof. Intuitively, as y becomes large, the function $c(\theta)$ becomes very flat. Multiplying by $\phi(y-\theta)$ and integrating essentially samples the function $c(\theta)$ at y.

To prove this assertion, apply the asymptotic approximation

$$
\operatorname{erfc}(z) \sim \frac{\exp(-z^2)}{\sqrt{\pi}z}
$$

to equation (2). With a little rearrangement, the arguments inside the exponential functions become zero and the integral reduces to

$$\frac{1}{2\sqrt{2\pi}\sigma}\left\{\frac{\exp(0)}{\sqrt{\pi}\left(\frac{1-iy}{\sqrt{2}\sigma}\right)}+\frac{\exp(0)}{\sqrt{\pi}\left(\frac{1+iy}{\sqrt{2}\sigma}\right)}\right\}=\frac{1}{\pi(1+y^2)}.$$

□

Next we apply the asymptotic series for erfc to equation (3).

Claim 5. *As $y \to \infty$,*

$$\int_{-\infty}^{\infty}\theta\phi(y-\theta)\,c(\theta)\,d\theta \sim \frac{y}{\pi}\left\{\frac{(1+y^2)^2+\sigma^2(y^2-3)}{(1+y^2)^3}\right\}.$$

Proof. Here we apply the two-term asymptotic approximation

$$\operatorname{erfc}(z) \sim \frac{\exp(-z^2)}{\sqrt{\pi}z}\left(1-\frac{1}{2z^2}\right).$$

As before, the arguments of the exponential functions become zero and the integral reduces to

$$\frac{1}{2\pi}\left\{\frac{1}{1-iy}\left(1-\left(\frac{\sigma}{1-iy}\right)^2\right)-\frac{1}{1+iy}\left(1-\left(\frac{\sigma}{1+iy}\right)^2\right)\right\}$$

which further reduces to

$$\frac{y}{\pi}\left\{\frac{(1+y^2)^2+\sigma^2(y^2-3)}{(1+y^2)^3}\right\}.$$

□

Consider a single sample y from a normal(θ, σ^2) distribution where θ has a conjugate normal(0, τ^2) prior. It is well-known that the posterior distribution on θ has mean

$$\frac{\tau^2}{\tau^2+\sigma^2}y.$$

Claim 6. *If we use a Cauchy(0, 1) prior rather than a normal(0, τ^2) prior on θ above, the posterior mean of θ is*

$$y - \mathcal{O}\left(\frac{1}{y}\right)$$

as $y \to \infty$.

Proof. Simply take the ratio of the results of Claim 5 and Claim 4. □

Next consider taking multiple samples y_i from a normal(θ, σ^2) distribution. Denote the sample mean of the y_i values by $\overline{y}$. We examine the posterior mean of θ under normal and Cauchy priors as the number of samples n increases.

With a normal(0, τ^2) prior on θ, the posterior mean of θ after observing n samples with mean $\overline{y}$ is

$$\frac{1}{1+\frac{\sigma^2}{n\tau^2}}\overline{y} = \left(1 - \frac{\sigma^2}{n\tau^2} + \mathcal{O}\left(\frac{1}{n^2}\right)\right)\overline{y}.$$

Claim 7. *If θ has a Cauchy(0, 1) prior, then the posterior mean of θ after sampling n values with sample mean $\overline{y}$ is*

$$\overline{y} + \frac{(\overline{y}^2-3)\overline{y}}{(1+\overline{y}^2)^2}\frac{\sigma^2}{n} + \mathcal{O}\left(\frac{1}{n^2}\right).$$

Proof. Observing n values from a normal(θ, σ^2) distribution is the same as observing one value $\overline{y}$ from a normal(θ, σ^2/n) distribution. The claim can be established analogous to Claim 5 letting $n \to \infty$ rather than $y \to \infty$. □

The rate at which the posterior mean converges to $\overline{y}$ depends on τ in the case of the normal prior and on $\overline{y}$ in the case of the Cauchy prior. For any value of τ, the convergence is faster under the Cauchy prior for sufficiently large values of $\overline{y}$.

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