Determining Days of the Week

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Ever wanted to be able find the day of the week a day will fall on (or fell on) without a calendar? It is possible to do this in one's head, and with a little bit of practice it can be done quickly.

NB: These notes were written for a class I taught in the 1990s, i.e. in the 20th century. The method still works, but if you're learning the method for the first time in the 21st century, it would be easier to adjust the method slightly to make it easier to use for 21st century dates. More on that here:

`https://www.johndcook.com/blog/2022/05/07/day-of-the-week/`

1 How the Method Works

The following method works for any day in the twentieth century. It can easily be modified for days in other centuries.

1. Take the last two digits of the year and add the number of times four will go into that number.
2. Add a number corresponding to the month.
3. Add the day of the month.
4. Find the remainder when the sum is divided by seven.
5. The day of the week corresponds to the remainder as follows: Sunday = 1, Monday = 2, . . . , Friday = 6, Saturday = 0.

The numbers for the months are given below.

Month	Number	Month	Number
Jan	1	July	0
Feb	4	Aug	3
March	4	Sept	6
April	0	Oct	1
May	2	Nov	4
June	5	Dec	6

For example, suppose you want to find out what day of the week Pearl Harbor was attacked, December 7, 1941. The last two digits are 41, and 4 goes into 41 ten times, so after the first step we have 51. Adding 6 for December and 7 for the day of the month gives 64, which leaves a remainder of 1. Thus December 7, 1941 was a Sunday.

There is one exception to the above rule. If a day falls in January or February of leap year, subtract one. For example, to find the day of the week for February 14, 1964 one would calculate as follows: $64+16+4+14-1 = 97$, which leaves a remainder of 6, and so February 14 was on a Friday in 1964.

By the way, a year is a leap year if it is divisible by 4, except that years divisible by 100 are leaps years only if they are also divisible by 400. For example, 1900 was not a leap year, but 2000 will be.

For years in the 21st century, subtract one. For years in the 19th century, add two.

2 Why the Method Works

We begin by introducing a little notation. We say that two numbers are congruent mod m if they have the same remainder when divided by m. We write

$$a \equiv b \bmod m$$

if a is congruent to b mod m. For example,

$$23 \equiv 37 \pmod 7$$

because both leave a remainder of two when divided by seven. The important thing for our purposes is that if

$$a \equiv c \pmod m$$

and

$$b \equiv d \pmod m$$

then

$$a + b \equiv c + d \pmod m$$

and

$$ab \equiv cd \pmod m.$$

The other notation we will use is $\lfloor x \rfloor$ to denote the largest integer less than or equal to x. For example, $\lfloor 3.14 \rfloor = 3$ and $\lfloor 6 \rfloor = 6$.

January 1, 1900 was a Monday. We could find the day of the week for any day after that by counting the number of days since then and seeing what the remainder is when the number is divided by seven. For example, December 27, 1950 is 18,622 days after January 1, 1900. Since

$$18,622 = 2660 \times 7 + 2,$$

18,622 days later falls on the same day of the week as 2 days later. So December 27, 1950 was on a Wednesday.

We'll call this the brute force method: it works, but it's hardly easy. We're doing more work than we have to because we do not need to actually find the number of days since the beginning of the century, but the *remainder* when this number is divided by seven. The properties of modular arithmetic allow us to find the remainder when a number is divided by seven without having to find the number itself.

If the year $1900 + n$ is not a leap year, then there have been

$$365 \times n + \lfloor n/4 \rfloor$$

days since the beginning of the century, $365 \times n$ being the number of ordinary days and $\lfloor n/4 \rfloor$ being the number of leap days. Since $365 \equiv 1 \pmod 7$,

$$365n + \lfloor n/4 \rfloor \equiv n + \lfloor n/4 \rfloor \pmod 7.$$

The reason we have to subtract 1 whenever we have a day in January or February of a leap year is that when we add in $\lfloor n/4 \rfloor$ we add in a leap day that hasn't happened yet.

If we add 1 for January and 1 for the first of the month, our method correctly tells us the day of the week for January 1 of any year. How do the month numbers come in? Since February 1 comes 31 days after January 1 each year, and $31 \equiv 3 \pmod 7$, the first of February comes three days later in the week than the first of January. Since the number for January is 1, the number for February is $1 + 3 = 4$. Since February has 28 days (leap days are accounted for separately), and $28 \equiv 0 \pmod 7$, the first of March occurs on the same day as the first of February and so the numbers of February and March are the same. The rest of the numbers are found similary.

Dates in the 21st century can be handled as so many years past 1900. For example, 2047 is 147 years past 1900 and so we could start with 147 and go on. But this gets cumbersome because we're dealing with larger numbers. If we were to apply our method the year 2000, using zero as the starting point instead of 100, we would get Sunday as the first day of 2000. But New Years Day 2000 should be Saturday. So if we treat years in the 21st century just like they were years in the 20th century, but remember to subtract one at the end, everything will work out correctly. The same reasoning can be used to find how to correct for other centuries.

3 Short Cuts

We can make the method faster by taking even more remainders by 7 along the way. For example, to find the day for January 28, 1933 we could add $33 + 8 + 1$, no need to add the 28 since it won't change the remainder. Becoming familiar with multiples of seven will help speed up this process.

One word of caution. We cannot take the remainder by seven before finding $\lfloor n/4 \rfloor$ because it is not true that if $a \equiv b \pmod 7$ then $\lfloor a/4 \rfloor \equiv \lfloor b/4 \rfloor \pmod 7$. For example, let $a = 4$ and $b = 5$.

However, if a year $1900 + n$ is divisible by seven and by four, *i.e.* divisible by 28, then $n + \lfloor n/4 \rfloor \equiv 0 \pmod 7$ and so there is no need to calculate the year part. For example, to find the day of the week for days in 1984, the first step in the process can be skipped. The same is true for 1928 and 1956 since 28, 56 and 84 are all multiples of 28. Also, we can subtract multiples of 28 before we start. For example, in 1995, we could start with 11 instead of 95 since $95 = 84 + 11$. Mathematically, this is because if k is an integer,

$$28k + n + \lfloor (28k + n)/4 \rfloor \equiv n + \lfloor n/4 \rfloor \pmod{7}.$$