Solution to the Recurrence Relation

1 Problem Statement

Solve the linear non-homogeneous recurrence relation

$$u_{n+2} = 6u_{n+1} - u_n + 2, \quad n \geq 0$$

with initial conditions

$$u_0 = 0, \quad u_1 = 3.$$

2 Homogeneous Solution

The associated homogeneous equation is

$$u_{n+2} - 6u_{n+1} + u_n = 0.$$

The characteristic equation is

$$r^2 - 6r + 1 = 0.$$

The roots are

$$r = \frac{6 \pm \sqrt{36-4}}{2} = \frac{6 \pm \sqrt{32}}{2} = \frac{6 \pm 4\sqrt{2}}{2} = 3 \pm 2\sqrt{2}.$$

Let $r_1 = 3 + 2\sqrt{2}$ and $r_2 = 3 - 2\sqrt{2}$. The homogeneous solution is

$$u_n^{(h)} = Ar_1^n + Br_2^n,$$

where A and B are constants to be determined.

3 Particular Solution

Since the non-homogeneous term is a constant (+2), we try a constant particular solution $u_n^{(p)} = C$. Substitute into the recurrence:

$$C = 6C - C + 2 \implies C = 5C + 2 \implies -4C = 2 \implies C = -\frac{1}{2}.$$

Thus,

$$u_n^{(p)} = -\frac{1}{2}.$$

4 General Solution

The general solution is the sum of the homogeneous and particular solutions:

$$u_n = A(3 + 2\sqrt{2})^n + B(3 - 2\sqrt{2})^n - \frac{1}{2}.$$

5 Applying Initial Conditions

Using $u_0 = 0$:

$$A + B - \frac{1}{2} = 0 \implies A + B = \frac{1}{2}. \tag{1}$$

Using $u_1 = 3$:

$$A(3 + 2\sqrt{2}) + B(3 - 2\sqrt{2}) - \frac{1}{2} = 3 \implies A(3 + 2\sqrt{2}) + B(3 - 2\sqrt{2}) = \frac{7}{2}. \tag{2}$$

Solving the system (1)–(2) yields

$$A = \frac{1 + \sqrt{2}}{4}, \quad B = \frac{1 - \sqrt{2}}{4}.$$

6 Closed-Form Solution

The particular solution satisfying the recurrence and initial conditions is

$$\boxed{u_n = \frac{1 + \sqrt{2}}{4}(3 + 2\sqrt{2})^n + \frac{1 - \sqrt{2}}{4}(3 - 2\sqrt{2})^n - \frac{1}{2}.}$$