Illustrating the error in the delta method

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1 Introduction

This note examines the error in the delta method approximations to $\mathrm{E}(g(X))$ and $\mathrm{Var}(g(X))$ where $X \sim \mathrm{Normal}(\mu, \sigma^2)$ and $g(x) = \exp(x)/(1+\exp(x))$, the inverse of the logit function.

The delta method is based on the approximation

$$g(x) \approx g(\mu) + g'(\mu)(x - \mu)$$

and leads to the approximations

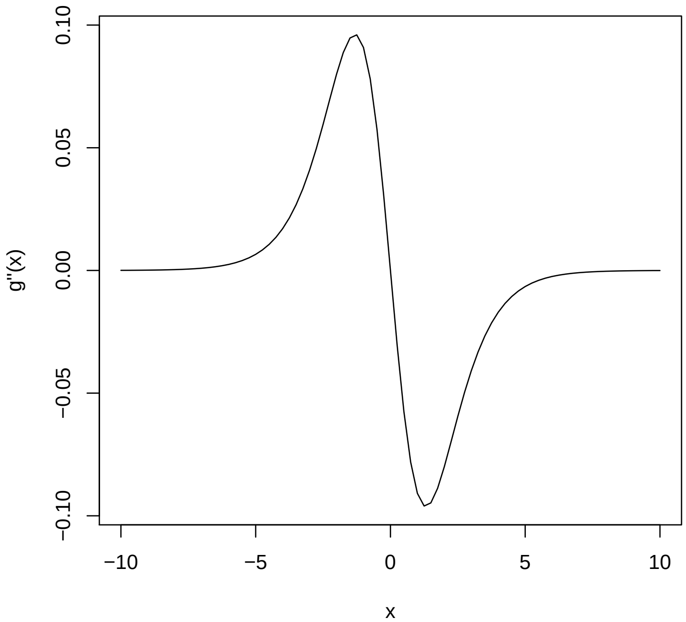
$$\mathrm{E}(g(X)) \approx g(\mathrm{E}(X)) = \frac{\exp(\mu)}{1 + \exp(\mu)}$$

and

$$\mathrm{Var}(g(X)) \approx (g'(\mathrm{E}(X)))^2 \mathrm{Var}(X) = \frac{\sigma^2 \exp(\mu)}{(1 + \exp(\mu))^2}.$$

In general, the delta method is most accurate when g is nearly linear, especially near $\mathrm{E}(X)$, and when $\mathrm{Var}(X)$ is small. We will see whether this hold up for our example. We compare the delta approximations to the exact values computed numerically varying μ and σ separately.

To see where the function g is most nearly linear, let's plot its second derivative.



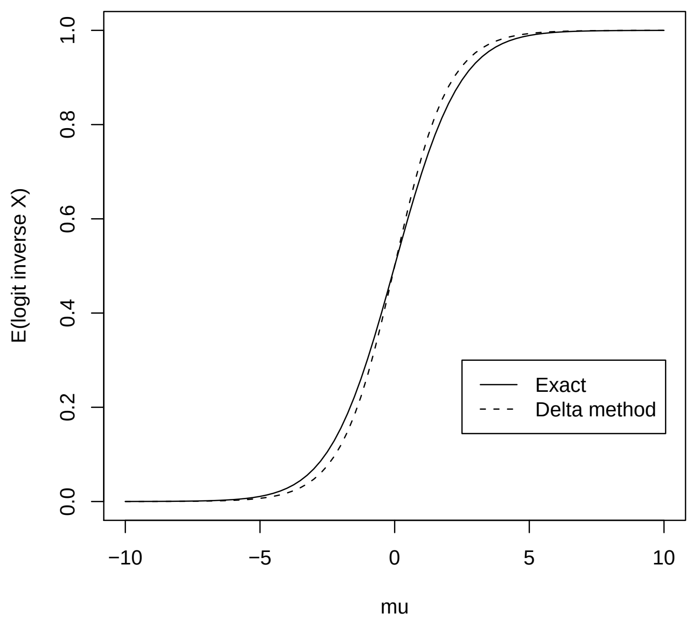
This suggests the delta method would be most accurate when $|\mu|$ is large. It also suggests the metho may work well for μ near zero, though small change in μ may make a big difference. Let's see how reality corresponds to our predictions.

2 Varying μ

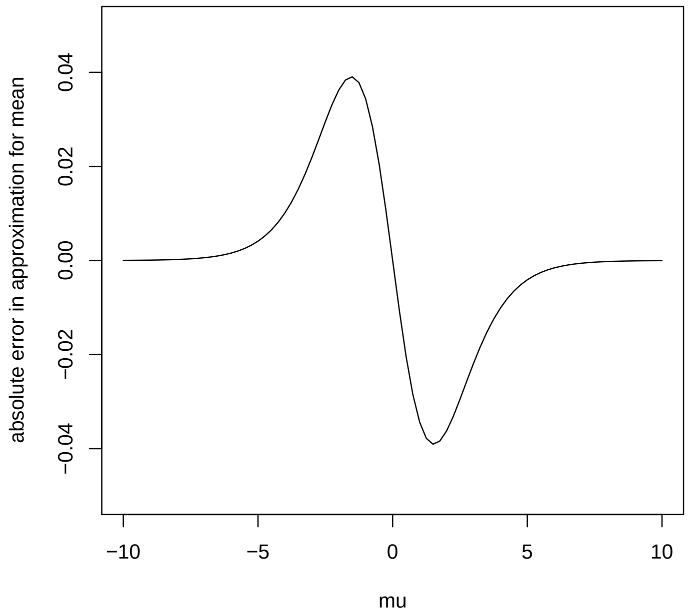
In this section $X \sim \text{Normal}(\mu, 1)$ and we let the mean μ vary from -10 to 10.

2.1 Estimating the mean

The figure below shows the mean of $g(X)$ as a function of the mean of X.



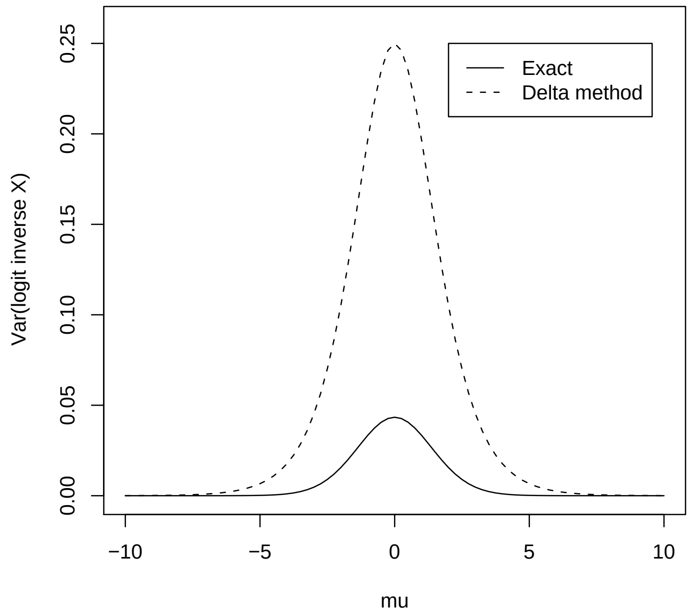
The following graph looks at the difference between the two curves above to show the absolute error.



The absolute error curve looks remarkably like the second derivative curve, confirming that in this example, the delta method is more accurate when g is nearly flat near μ.

2.2 Estimating the variance

The error in estimating the variance is fairly large for moderate values of μ. As expected, the error decreases as $|\mu|$ increases.

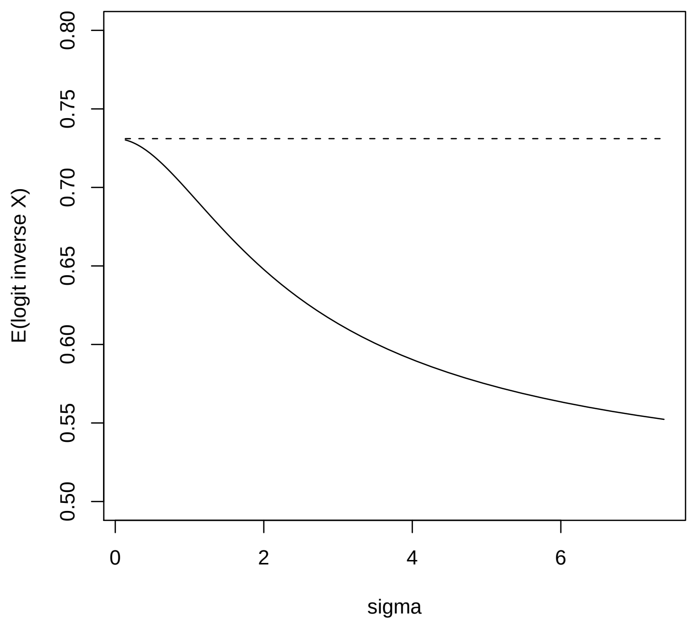


3 Varying σ

In this section, we fix $\mu = 1$ and vary σ.

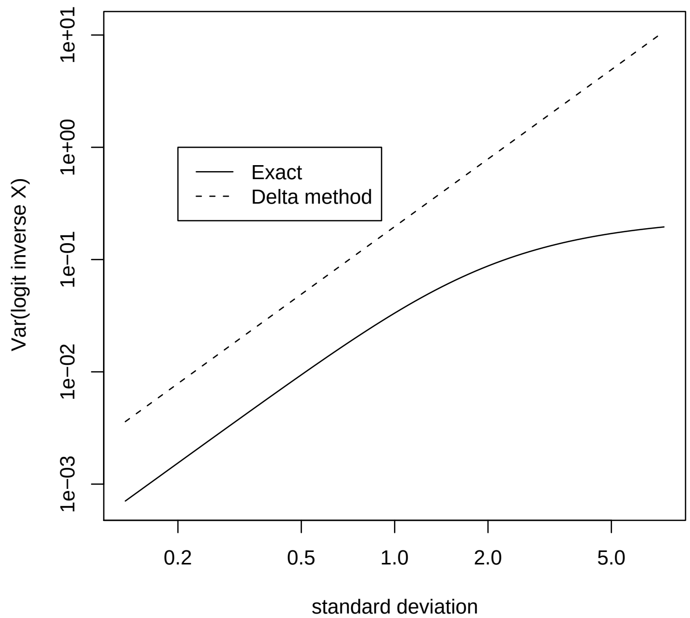
3.1 Estimating the mean

The delta method estimate of the mean does not depend on the variance X and hence the estimate is constant. However, $\mathrm{E}(g(X))$ increases with σ and so the error increases with σ, as expected.



3.2 Estimating the variance

The quality of the estimate of $\mathrm{Var}(g(X))$ depends exponentially on $\mathrm{Var}(X)$ and so we plot the estimate and the true value on a log-log scale. For small values of σ, the delta method approximation is quite good. For larger values of σ, the quality is bad and degrades exponentially.



This document is available at `http://www.johndcook.com/delta_method.pdf`.