Abstract

This note surveys results for computing the inequality probability

\[ P(X > Y) \]

in closed form where \( X \) and \( Y \) are independent continuous random variables. Distribution families discussed include

- normal,
- Cauchy,
- gamma,
- inverse gamma,
- Lévy,
- folded normal, and
- beta.

Mixture distributions are also discussed.

1 Normal and Cauchy random variables

The probability

\[ P(X > Y) \]
can be evaluated simply when $X$ and $Y$ are either both normal or both Cauchy random variables. These probabilities are derived in [1].

If $X$ is normal with mean $\mu_X$ and variance $\sigma^2_X$, and $Y$ is normal with mean $\mu_Y$ and variance $\sigma^2_Y$, then

$$P(X > Y) = \Phi \left( \frac{\mu_X - \mu_Y}{(\sigma^2_X + \sigma^2_Y)^{1/2}} \right)$$

where $\Phi(x)$ is the CDF of a standard normal random variable.

If $X$ is Cauchy with location $\mu_X$ and scale $\sigma_X$, and $Y$ is Cauchy with location $\mu_Y$ and scale $\sigma_Y$, then

$$P(X > Y) = \frac{1}{2} + \frac{1}{\pi} \tan^{-1} \left( \frac{\mu_X - \mu_Y}{\sigma_X + \sigma_Y} \right).$$

2 Gamma, inverse gamma, and Lévy random variables

If $X$ has a gamma($\alpha_X, \beta_X$) distribution and $Y$ has a gamma($\alpha_Y, \beta_Y$) distribution then

$$P(X > Y) = P \left( B < \frac{\beta_Y}{\beta_X + \beta_Y} \right) = I_{\frac{\beta_Y}{\beta_X + \beta_Y}}(\alpha_Y, \alpha_X)$$

where $B \sim \text{beta}(\alpha_Y, \alpha_X)$ and $I_a(a,b)$ is the regularized incomplete beta function. This result first appeared in [1].

If $X$ has an inverse gamma distribution, $1/X$ has a gamma distribution. Therefore inverse gamma inequality probabilities can be reduced to gamma inequality probabilities since

$$P(X > Y) = P(1/Y > 1/X).$$

A Lévy distribution with scale $c$ is the same as an inverse gamma distribution with shape $1/2$ and scale $c/2$, and so inequalities with Lévy distributions are a special case of inequalities with inverse gamma distributions.

(There are multiple conventions for parameterizing gamma and inverse gamma distributions. The results here assume the density of a gamma($\alpha, \beta$) random variable is proportional to $x^{\alpha-1} \exp(-x/\beta)$. Also, the density of an inverse gamma($\alpha, \beta$) random variable is proportional to $x^{-\alpha-1} \exp(-\beta/x)$.)
3  Folded normal random variables

A folded normal random variable is the absolute value of a normal random variable. Therefore folded normal random inequalities reduce to computing $P(|X| > |Y|)$ where $X$ and $Y$ are normal.

Suppose $X$ is normal($\mu_X, \sigma_X^2$) and $Y$ is normal($\mu_Y, \sigma_Y^2$). If $\sigma_X^2 = \sigma_Y^2$ then $P(|X| > |Y|)$ can be computed as follows. Let $U = X + Y$ and $V = X - Y$. Then

\[ U \sim N(\mu_U, \sigma_U^2) = N(\mu_X + \mu_Y, \sigma_X + \sigma_Y) \]
\[ V \sim N(\mu_V, \sigma_V^2) = N(\mu_X - \mu_Y, \sigma_X + \sigma_Y) \]

When $\sigma_X^2 = \sigma_Y^2$, we have

\[ \Pr(|X| > |Y|) = \Phi\left(\frac{-\mu_U}{\sigma_U}\right)\Phi\left(\frac{-\mu_V}{\sigma_V}\right) + \Phi\left(\frac{\mu_U}{\sigma_U}\right)\Phi\left(\frac{\mu_V}{\sigma_V}\right). \tag{1} \]

See [4] for a derivation of the above result as well as a method for evaluating $P(|X| > |Y|)$ when the variances of $X$ and $Y$ are not equal.

4  Beta random variables

In general $P(X > Y)$ cannot be evaluated in closed form if $X$ and $Y$ have beta distributions. However, many special cases do have closed form expressions. Let $X \sim \text{beta}(a, b)$ and $Y \sim \text{beta}(c, d)$. Define

\[ g(a, b, c, d) = P(X > Y). \]

The function $g(a, b, c, d)$ can be evaluated in closed form if

- one of the four parameters $a$, $b$, $c$, or $d$ is an integer,
- the fractional parts of the parameters sum to 1, or
- $a + b$ and $c + d$ are positive integers.

The symmetries

\[ g(a, b, c, d) = g(d, c, b, a) = g(d, b, c, a) = 1 - g(c, d, a, b) \]
are developed in [1]. For the 24 permutations of the 4 arguments, there are at most six different values of $g$. These are $g(a, b, c, d)$, $g(a, b, d, c)$, $g(a, c, d, b)$ and their complementary probabilities.

If we define

$$h(a, b, c, d) = \frac{B(a + c, b + d)}{B(a, b)B(c, d)}$$

(2)

$$= \frac{\Gamma(a + c)\Gamma(b + d)\Gamma(a + b)\Gamma(c + d)}{\Gamma(a)\Gamma(b)\Gamma(c)\Gamma(d)\Gamma(a + b + c + d)}.$$  

then [1] shows that the following recurrence relations hold:

$$g(a + 1, b, c, d) = g(a, b, c, d) + h(a, b, c, d)/a$$
$$g(a, b + 1, c, d) = g(a, b, c, d) - h(a, b, c, d)/b$$
$$g(a, b, c + 1, d) = g(a, b, c, d) - h(a, b, c, d)/c$$
$$g(a, b, c, d + 1) = g(a, b, c, d) + h(a, b, c, d)/d.$$  

The identity

$$g(a, b, c, 1) = \frac{\Gamma(a + b)\Gamma(a + c)}{\Gamma(a + b + c)\Gamma(a)}$$

is derived in [2]. The symmetries and recurrence relationships above can be used to bootstrap this result into a closed form evaluation of $g$ when any one of its arguments is an integer.

If $a + b + c + d = 1$, then [2] shows that

$$g(a, b, c, d) = \frac{\sin(\pi a)\sin(\pi b)\sin(\pi d)}{\sin(\pi(a + b))\sin(\pi(b + d))}.$$  

Symmetry and recurrence relationships can extend this result to compute $g$ for any arguments whose fractional parts sum to 1.

Finally, if $a + b = c + d = 1$ and $a + c \neq 1$ then

$$g(a, b, c, d) = \frac{\Gamma(a + c)\Gamma(b + d)\sin(\pi a 1)\sin(\pi b 2)\left(c(\psi(c) - \psi(1 - a))\right)}{a + c - 1}$$

where $\psi$ is the digamma function. As before, this result can be extended by using symmetry and recurrence relationships.

For more information regarding beta inequalities, see [2] and [3].
5 Mixture distributions

If $P(X > Y)$ can be computed in closed form for distributions $X$ and $Y$ from a particular family, then it can also be computed in closed form for mixtures of distributions from that family.

Suppose

$$f_X = \sum_{i=1}^{n} \lambda_i f_{X_i}$$

and

$$f_Y = \sum_{j=1}^{m} \theta_j f_{Y_j}$$

where $\lambda_i \geq 0$, $\theta_i \geq 0$, and

$$\sum_{i=1}^{n} \lambda_i = \sum_{j=1}^{m} \theta_j = 1.$$

Then

$$P(X > Y) = \sum_{i=1}^{n} \sum_{j=1}^{m} \lambda_i \theta_j P(X_i > Y_j).$$

References


