Inequality Probabilities for Folded Normal Random Variables

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Abstract

This note explains how to calculate the probability

$$\Pr(|X|>|Y|)$$

(1)

for normal random variables $X \sim N(\mu_X, \sigma_X^2)$ and $Y \sim N(\mu_Y, \sigma_Y^2)$. A random variable formed by taking the absolute value of a normal random variable is known as a folded normal random variable.

When $\sigma_X = \sigma_Y$, (1) can be evaluated simply using Equation (3) below. When $\sigma_X \neq \sigma_Y$, (1) can be reduced to a well-known problem using Equation (4).

1 Removing absolute values

To make the problem (1) easier to work with, we restate the problem in a form that does not involve absolute values. We begin by noting that the set of points

$$\{|x|>|y|\}$$

is bounded by the lines $x+y=0$ and $x-y=0$ and can thus be written as

$$\{x+y>0 \land x-y>0\} \cup \{x+y<0 \land x-y<0\}.$$
It follows that
\[
\Pr(|X| > |Y|) = \Pr(X + Y > 0 \land X - Y > 0) + \Pr(X + Y < 0 \land X - Y < 0).
\]

Now define
\[
U = X + Y \\
V = X - Y
\]
and so
\[
U \sim N(\mu_U, \sigma_U^2) \\
V \sim N(\mu_V, \sigma_V^2)
\]
where \(\mu_Y = \mu_x + \mu_y, \mu_U = \mu_x + \mu_y,\) and \(\sigma_U^2 = \sigma_V^2 = \sigma_X^2 + \sigma_Y^2.\) We now have
\[
\Pr(|X| > |Y|) = \Pr(U > 0 \land V > 0) + \Pr(U < 0 \land V < 0). \quad (2)
\]

## 2 Joint probabilities

We now move on to calculating each of the inequalities on the right-hand side of Equation (2). First note that
\[
\Pr(U > 0 \land V > 0) = \Pr(U - \mu_U > -\mu_U \land V - \mu_V > -\mu_V)
\]
\[
= \Pr \left( \frac{U - \mu_U}{\sigma_U} > \frac{-\mu_U}{\sigma_U} \land \frac{V - \mu_V}{\sigma_V} > \frac{-\mu_V}{\sigma_V} \right)
\]
\[
= \Pr \left( Z_1 > \frac{-\mu_U}{\sigma_U} \land Z_2 > \frac{-\mu_V}{\sigma_V} \right)
\]
where \(Z_1 = (U - \mu_U)/\sigma_U\) and \(Z_2 = (V - \mu_V)/\sigma_V\) are standard normal random variables. Similarly,
\[
\Pr(U < 0 \land V < 0) = \Pr \left( Z_1 < \frac{\mu_U}{\sigma_U} \land Z_2 < \frac{\mu_V}{\sigma_V} \right).
\]
The random variables \(Z_1\) and \(Z_2\) have correlation
\[
\frac{\sigma_X^2 - \sigma_Y^2}{\sigma_X^2 + \sigma_Y^2}.
\]
When $\sigma_X^2 = \sigma_Y^2$, $Z_1$ and $Z_2$ are uncorrelated and we have

$$\Pr(|X| > |Y|) = \Phi \left( -\frac{\mu_U}{\sigma_U} \right) \Phi \left( -\frac{\mu_V}{\sigma_V} \right) + \Phi \left( \frac{\mu_U}{\sigma_U} \right) \Phi \left( \frac{\mu_V}{\sigma_V} \right)$$

(3)

where $\Phi(x)$ is the CDF of a standard normal random variable.

When $\sigma_X^2 \neq \sigma_Y^2$, Equation (3) does not hold. However, in that case we may still evaluate $\Pr(|X| > |Y|)$ as

$$\Pr \left( Z_1 > -\frac{\mu_U}{\sigma_U} \wedge Z_2 > -\frac{\mu_V}{\sigma_V} \right) + \Pr \left( Z_1 < \frac{\mu_U}{\sigma_U} \wedge Z_2 < \frac{\mu_V}{\sigma_V} \right)$$

(4)

Expression (4) cannot be evaluated in closed form. However, it does reduce to a known problem: evaluating rectangular probabilities for a bivariate normal random variable. These can be reduced to a one-dimensional integral that can be evaluated numerically. See “Numerical Computation of Rectangular Bivariate and Trivariate Normal and $t$ Probabilities” by Alan Genz, Statistics and Computing, 14 (2004), pp. 151-160.