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Inequality Probabilities for Folded Normal Random Variables

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Abstract

This note explains how to calculate the probability

$$\Pr(|X| > |Y|) \tag{1}$$

for normal random variables $X \sim N(\mu_X, \sigma_X^2)$ and $Y \sim N(\mu_Y, \sigma_Y^2)$. A random variable formed by taking the absolute value of a normal random variable is known as a *folded* normal random variable.

When $\sigma_X = \sigma_Y$, (1) can be evaluated simply using Equation (3) below. When $\sigma_X \neq \sigma_Y$, (1) can be reduced to a well-known problem using Equation (4).

1 Removing absolute values

To make the problem (1) easier to work with, we restate the problem in a form that does not involve absolute values. We begin by noting that the set of points

$$\{|x| > |y|\}$$

is bounded by the lines $x + y = 0$ and $x - y = 0$ and can thus be written as

$$\{x + y > 0 \ \wedge \ x - y > 0\} \cup \{x + y < 0 \ \wedge \ x - y < 0\}.$$

It follows that

$$\begin{aligned}\Pr(|X| > |Y|) &= \Pr(X + Y > 0 \ \wedge \ X - Y > 0) \\ &+ \Pr(X + Y < 0 \ \wedge \ X - Y < 0).\end{aligned}$$

Now define

$$\begin{aligned}U &= X + Y \\ V &= X - Y\end{aligned}$$

and so

$$\begin{aligned}U &\sim N(\mu_U, \sigma_U^2) \\ V &\sim N(\mu_V, \sigma_V^2)\end{aligned}$$

where $\mu_Y = \mu_x + \mu_y$, $\mu_U = \mu_x + \mu_y$, and $\sigma_U^2 = \sigma_V^2 = \sigma_X^2 + \sigma_Y^2$. We now have

$$\Pr(|X| > |Y|) = \Pr(U > 0 \ \wedge \ V > 0) + \Pr(U < 0 \ \wedge \ V < 0). \tag{2}$$

2 Joint probabilities

We now move on to calculating each of the inequalities on the right-hand side of Equation (2). First note that

$$\begin{aligned}\Pr(U > 0 \ \wedge \ V > 0) &= \Pr(U - \mu_U > -\mu_U \ \wedge \ V - \mu_V > -\mu_V) \\ &= \Pr\left(\frac{U - \mu_U}{\sigma_U} > -\frac{\mu_U}{\sigma_U} \ \wedge \ \frac{V - \mu_V}{\sigma_V} > -\frac{\mu_V}{\sigma_V}\right) \\ &= \Pr\left(Z_1 > -\frac{\mu_U}{\sigma_U} \ \wedge \ Z_2 > -\frac{\mu_V}{\sigma_V}\right)\end{aligned}$$

where $Z_1 = (U - \mu_U)/\sigma_U$ and $Z_2 = (V - \mu_V)/\sigma_V$ are standard normal random variables. Similarly,

$$\Pr(U < 0 \ \wedge \ V < 0) = \Pr\left(Z_1 < \frac{\mu_U}{\sigma_U} \ \wedge \ Z_2 < \frac{\mu_V}{\sigma_V}\right).$$

The random variables Z_1 and Z_2 have correlation

$$\frac{\sigma_X^2 - \sigma_Y^2}{\sigma_X^2 + \sigma_Y^2}.$$

When $\sigma_X^2 = \sigma_Y^2$, Z_1 and Z_2 are uncorrelated and we have

$$\Pr(|X| > |Y|) = \Phi\left(-\frac{\mu_U}{\sigma_U}\right)\Phi\left(-\frac{\mu_V}{\sigma_V}\right) + \Phi\left(\frac{\mu_U}{\sigma_U}\right)\Phi\left(\frac{\mu_V}{\sigma_V}\right) \tag{3}$$

where $\Phi(x)$ is the CDF of a standard normal random variable.

When $\sigma_X^2 \neq \sigma_Y^2$, Equation (3) does not hold. However, in that case we may still evaluate $\Pr(|X| > |Y|)$ as

$$\Pr\left(Z_1 > -\frac{\mu_U}{\sigma_U} \ \wedge\ Z_2 > -\frac{\mu_V}{\sigma_V}\right) + \Pr\left(Z_1 < \frac{\mu_U}{\sigma_U} \ \wedge\ Z_2 < \frac{\mu_V}{\sigma_V}\right). \tag{4}$$

Expression (4) cannot be evaluated in closed form. However, it does reduce to a known problem: evaluating rectangular probabilities for a bivariate normal random variable. These can be reduced to a one-dimensional integral that can be evaluated numerically. See "Numerical Computation of Rectangular Bivariate and Trivariate Normal and t Probabilities" by Alan Genz, Statistics and Computing, 14 (2004), pp. 151-160.