Defining ODEs for Holonomic Functions in Abramowitz & Stegun

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1 Holonomic functions

Holonomic functions are solutions to linear differential equations with polynomial coefficients.

$$p_n(x)y^{(n)} + \cdots + p_0(x)y = 0$$

This report shows that many special functions are holonomic by giving their corresponding differential equations.

The definition of holonomic allows differential equations of any order, but the differential equations in this report are all second order with the following exceptions.

The functions exp and log satisfy first order equations, and the functions Si and Ci satisfy third order equations. The generalized hypergeometric function ${}_pF_q$ satisfies a differential equation of order $\max(p, q+1)$, and so can have any order. However, the most common hypergeometic functions, ${}_1F_1$ and ${}_2F_1$, satisfy a second order differential equation.

2 Non-holonomic functions

The functions tan, cot, sec, csc, tanh, coth, sech, and csch are *not* holonomic. A linear ODE with polynomial coefficients can only have singularities at the zeros of of the leading coefficient p_n. But $\tan x$ has poles at $x = \frac{\pi}{2} + k\pi$ for every integer k — infinitely many in the finite plane — so no such equation can exist. The same argument shows that cot, sec and csc cannot be holonomic.

Considering poles along the imaginary axis shows that tanh, coth, sech, and csch cannot be holonomic.

However, the **inverses** of the functions above *are* holonomic.

The gamma function and related functions, such as digamma and beta, also have infinitely many singularities and so cannot be holonomic. The same is true of elliptic functions.

Aside from these exceptions, all the special functions list in Abramowitz & Stegun are holonomic.

3 Elementary functions

Function	ODE	Conditions
$\exp(x)$	$y' - y = 0$	$y(0) = 1$
$\log(x)$	$xy' - 1 = 0$	$y(1) = 0$
$\sin(x)$	$y'' + y = 0$	$y(0) = 0,\ y'(0) = 1$
$\cos(x)$	$y'' + y = 0$	$y(0) = 1,\ y'(0) = 0$
$\sinh(x)$	$y'' - y = 0$	$y(0) = 0,\ y'(0) = 1$
$\cosh(x)$	$y'' - y = 0$	$y(0) = 1,\ y'(0) = 0$
$\arcsin(x)$	$(1 - x^2)y'' - xy' = 0$	$y(0) = 0,\ y'(0) = 1$
$\arccos(x)$	$(1 - x^2)y'' - xy' = 0$	$y(0) = \pi/2,\ y'(0) = -1$
$\arctan(x)$	$(1 + x^2)y'' + 2xy' = 0$	$y(0) = 0,\ y'(0) = 1$
$\operatorname{arcsinh}(x)$	$(1 + x^2)y'' + xy' = 0$	$y(0) = 0,\ y'(0) = 1$
$\operatorname{arccosh}(x)$ $(x > 1)$	$(x^2 - 1)y'' + xy' = 0$	$y(x) \sim \sqrt{2(x-1)}$ as $x \to 1^+$
$\operatorname{arctanh}(x)$	$(1 - x^2)y'' - 2xy' = 0$	$y(0) = 0,\ y'(0) = 1$

4 Exponential, logarithmic, sine/cosine integrals (A&S Ch. 5)

$$\mathrm{Ei}(x): \quad xy'' + (1-x)y' = 0, \qquad y(x) \sim \frac{e^x}{x}\ (x \to \infty), \quad y(x) - \ln x \to \gamma\ (x \to 0^+)$$

(the two independent solutions are 1 and $\mathrm{Ei}(x)$ itself; the log-behavior at 0 fixes which combination.)

$$\mathrm{li}(x) = \mathrm{Ei}(\ln x) \quad \text{(algebraic substitution of the above)}$$

$$\mathrm{Si}(x),\ \mathrm{Ci}(x): \quad xy''' + 2y'' + xy' = 0$$

- Si: $y(0) = 0,\ y'(0) = 1,\ y''(0) = 0$
- Ci: $y(x) - \ln x \to \gamma$ as $x \to 0^+$, and $y(x) \to 0$ as $x \to \infty$

5 Incomplete gamma function (fixed a)

Both $\gamma(a, x)$ and $\Gamma(a, x)$ satisfy the second-order, homogeneous, polynomial-coefficient equation

$$xy'' + (a + 1 - x)y' - (a - 1)y = 0,$$

with

- $\gamma(a,x)$: $y(x) \sim x^a/a$ as $x \to 0^+$
- $\Gamma(a,x)$: $y(x) \sim x^{a-1}e^{-x}$ as $x \to \infty$

6 Error function, Fresnel integrals, Dawson's function

$$\mathrm{erf}(x): \quad y'' + 2xy' = 0, \qquad y(0) = 0,\ y'(0) = \tfrac{2}{\sqrt{\pi}}$$

$$C(x), S(x)\ \text{(Fresnel)}: \quad y''' - \frac{1}{x}y'' + \pi^2 x^2 y' = 0$$

(from $C' = \cos(\frac{\pi}{2}x^2)$, etc.), with $C(0) = S(0) = 0$, $C'(0) = 1,\ S'(0) = 0$ (and analogously for S).

$$\text{Dawson's } F(x): \quad y'' + 2xy' + 2y = 0, \qquad y(0) = 0,\ y'(0) = 1$$

7 Legendre functions P_ν, Q_ν (general ν)

$$(1 - x^2)y'' - 2xy' + \nu(\nu+1)y = 0$$

- P_ν: $y(1) = 1$ (regular at $x = 1$)
- Q_ν: $y(x) \to 0$ as $x \to \infty$ (or, on $(-1,1)$, the solution singular at $x = 1$, normalized by its standard series)

8 Bessel, spherical Bessel, Airy, Struve, Kelvin functions

Bessel J_ν, Y_ν:

$$x^2y'' + xy' + (x^2 - \nu^2)y = 0$$

- J_ν: $y(x) \sim \dfrac{(x/2)^\nu}{\Gamma(\nu+1)}$ as $x \to 0^+$
- Y_ν: $y(x) \sim \sqrt{\dfrac{2}{\pi x}}\sin\Big(x - \dfrac{\nu\pi}{2} - \dfrac{\pi}{4}\Big)$ as $x \to \infty$

Modified Bessel I_ν, K_ν:

$$x^2y'' + xy' - (x^2 + \nu^2)y = 0$$

- I_ν: $y(x) \sim \dfrac{(x/2)^\nu}{\Gamma(\nu+1)}$ as $x \to 0^+$

- K_ν: $y(x) \sim \sqrt{\frac{\pi}{2x}} e^{-x}$ as $x \to \infty$

Spherical Bessel j_n, y_n**:**

$$x^2 y'' + 2xy' + [x^2 - n(n+1)]y = 0$$

- j_n: $y(x) \sim \frac{x^n}{(2n+1)!!}$ as $x \to 0$
- y_n: $y(x) \sim -\frac{(2n-1)!!}{x^{n+1}}$ as $x \to 0$

Airy Ai, Bi**:**

$$y'' - xy = 0$$

- Ai: $y(x) \to 0$ as $x \to +\infty$, with $\mathrm{Ai}(0) = \frac{1}{3^{2/3}\Gamma(2/3)}$
- Bi: $\mathrm{Bi}(0) = \frac{1}{3^{1/6}\Gamma(2/3)}, \quad \mathrm{Bi}'(0) = \frac{3^{1/6}}{\Gamma(1/3)}$

Struve H_ν**:**

$$x^2 y'' + xy' + (x^2 - \nu^2)y = \frac{4(x/2)^{\nu+1}}{\sqrt{\pi}\,\Gamma(\nu + \frac{1}{2})}$$

$$y(x) \sim \frac{(x/2)^{\nu+1}}{\Gamma(\frac{3}{2})\Gamma(\nu + \frac{3}{2})} \quad (x \to 0^+)$$

Kelvin $\mathrm{ber}_\nu, \mathrm{bei}_\nu$ (as $w = \mathrm{ber}_\nu + i\,\mathrm{bei}_\nu$):

$$x^2 w'' + xw' - (ix^2 + \nu^2)w = 0, \qquad w(x) \sim \frac{(xe^{3\pi i/4}/2)^\nu}{\Gamma(\nu+1)} \; (x \to 0^+)$$

$\mathrm{ker}_\nu, \mathrm{kei}_\nu$ satisfy the same ODE, normalized instead by decay as $x \to \infty$ (analogous to K_ν).

9 Confluent hypergeometric & Whittaker functions

$$M(a,b,x) = {}_1F_1(a;b;x): \quad xy'' + (b-x)y' - ay = 0, \qquad y(0) = 1$$

$$U(a,b,x): \text{ same ODE}, \qquad y(x) \sim x^{-a} \; (x \to \infty)$$

$$\text{Whittaker } M_{\kappa,\mu}, W_{\kappa,\mu}: \quad y'' + \left[-\frac{1}{4} + \frac{\kappa}{x} + \frac{1/4 - \mu^2}{x^2}\right] y = 0$$

- $M_{\kappa,\mu}$: $y(x) \sim x^{\mu+1/2}$ as $x \to 0^+$
- $W_{\kappa,\mu}$: $y(x) \sim e^{-x/2} x^\kappa$ as $x \to \infty$

10 Coulomb wave functions F_L, G_L

$$y'' + \left[1 - \frac{2\eta}{\rho} - \frac{L(L+1)}{\rho^2}\right] y = 0$$

- F_L: $y(\rho) \sim C_L(\eta)\rho^{L+1}$ as $\rho \to 0$
- G_L: $y(\rho) \sim \dfrac{\rho^{-L}}{(2L+1)C_L(\eta)}$ as $\rho \to 0$

11 Generalized hypergeometric ${}_pF_q$

With $\theta = x\,d/dx$:

$$[\theta(\theta + b_1 - 1)\cdots(\theta + b_q - 1) - x(\theta + a_1)\cdots(\theta + a_p)]\, y = 0$$

$$y(0) = 1$$

12 Mathieu, spheroidal wave, parabolic cylinder functions

Mathieu $\mathrm{ce}_n, \mathrm{se}_n$ (classical form; coefficients are holonomic-in-x but not literally polynomial — a genuinely polynomial-coefficient equation of higher order exists by closure but is unwieldy to write):

$$y'' + (a_n - 2q\cos 2x)y = 0$$

ce_n: even, period π or 2π, normalized so its Fourier expansion has leading coefficient 1; se_n: odd, likewise.

Spheroidal wave functions (this one genuinely has polynomial/rational coefficients):

$$(1 - x^2)y'' - 2xy' + \left[\lambda_{mn} - c^2x^2 - \frac{m^2}{1 - x^2}\right] y = 0$$

normalized as $P_n^m(x)$ when $c = 0$: $y(1)$ finite, matching Legendre normalization.

Parabolic cylinder (Weber) $U(a, x) = D_a(x)$**:**

$$y'' + \left(a + \frac{1}{2} - \frac{x^2}{4}\right) y = 0$$

$$y(0) = \frac{2^{a/2+1/4}\sqrt{\pi}}{\Gamma\left(\frac{a}{2} + \frac{3}{4}\right)}, \qquad y'(0) = -\frac{2^{a/2+3/4}\sqrt{\pi}}{\Gamma\left(\frac{a}{2} + \frac{1}{4}\right)}$$

13 Elliptic integrals (as functions of modulus k)

$$K(k): \quad k(1-k^2)y'' + (1-3k^2)y' - ky = 0, \qquad y(0) = \frac{\pi}{2}$$

$$E(k): \quad k(1-k^2)y'' + (1-k^2)y' + ky = 0, \qquad y(0) = \frac{\pi}{2},\ y'(0) = 0$$

Incomplete $F(\phi, k), E(\phi, k), \Pi(n; \phi, k)$ are more naturally treated as holonomic in $x = \sin\phi$ for fixed k, via $dF/dx = 1/\sqrt{(1-x^2)(1-k^2x^2)}$, giving a similar second-order equation in x.

14 Classical orthogonal polynomials (fixed degree n)

Polynomial	ODE	Normalization
Legendre P_n	$(1-x^2)y'' - 2xy' + n(n+1)y = 0$	$P_n(1) = 1$
Chebyshev T_n	$(1-x^2)y'' - xy' + n^2y = 0$	$T_n(1) = 1$
Chebyshev U_n	$(1-x^2)y'' - 3xy' + n(n+2)y = 0$	$U_n(1) = n+1$
Hermite H_n	$y'' - 2xy' + 2ny = 0$	leading coeff. 2^n
Laguerre L_n	$xy'' + (1-x)y' + ny = 0$	$L_n(0) = 1$
Gen. Laguerre $L_n^{(\alpha)}$	$xy'' + (\alpha+1-x)y' + ny = 0$	$L_n^{(\alpha)}(0) = \binom{n+\alpha}{n}$
Jacobi $P_n^{(\alpha,\beta)}$	$(1-x^2)y'' + [\beta-\alpha-(\alpha+\beta+2)x]y' + n(n+\alpha+\beta+1)y = 0$	$P_n^{(\alpha,\beta)}(1) = \binom{n+\alpha}{n}$
Gegenbauer $C_n^{(\lambda)}$	$(1-x^2)y'' - (2\lambda+1)xy' + n(n+2\lambda)y = 0$	$C_n^{(\lambda)}(1) = \binom{n+2\lambda-1}{n}$