Inverse Gamma Distribution

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Abstract

These notes write up some basic facts regarding the inverse gamma distribution, also called the inverted gamma distribution. In a sense this distribution is unnecessary: it has the same distribution as the reciprocal of a gamma distribution. However, a catalog of results for the inverse gamma distribution prevents having to repeatedly apply the transformation theorem in applications.

Here we derive the distribution of the inverse gamma, calculate its moments, and show that it is a conjugate prior for an exponential likelihood function.

1 Parameterizations

There are at least a couple common parameterizations of the gamma distribution. For our purposes, a gamma(α, β) distribution has density

$$f(x) = \frac{1}{\Gamma(\alpha)\beta^\alpha} x^{\alpha-1} \exp(-x/\beta)$$

for $x > 0$. With this parameterization, a gamma(α, β) distribution has mean $\alpha\beta$ and variance $\alpha\beta^2$.

Define the inverse gamma (IG) distribution to have the density

$$f(x) = \frac{\beta^\alpha}{\Gamma(\alpha)} x^{-\alpha-1} \exp(-\beta/x)$$

for $x > 0$.

2 Relation to the gamma distribution

With the above parameterizations, if X has a gamma(α, β) distribution then $Y = 1/X$ has an IG(α, $1/\beta$) distribution. To see this, apply the transformation theorem.

$$\begin{aligned} f_Y(y) &= f_X(1/y)\left|\frac{d}{dy}\,y^{-1}\right| \\ &= \frac{1}{\Gamma(\alpha)\beta^\alpha} y^{-\alpha+1}\exp(-1/\beta y)\,y^{-2} \\ &= \frac{(1/\beta)^\alpha}{\Gamma(\alpha)} y^{-\alpha-1}\exp(-(1/\beta)/y) \end{aligned}$$

3 Moments

Next we calculate the moments of $X \sim \text{IG}(\alpha, \beta)$. If $\alpha > n$,

$$\begin{aligned} E(X^n) &= \frac{\beta^\alpha}{\Gamma(\alpha)}\int_0^\infty x^n\,x^{-\alpha-1}\exp(-\beta/x)\,dx \\ &= \frac{\beta^\alpha}{\Gamma(\alpha)}\int_0^\infty x^{n-\alpha-1}\exp(-\beta/x)\,dx \\ &= \frac{\beta^\alpha}{\Gamma(\alpha)}\frac{\Gamma(\alpha-n)}{\beta^{\alpha-n}} \\ &= \frac{\beta^n\Gamma(\alpha-n)}{(\alpha-1)\cdots(\alpha-n)\Gamma(\alpha-n)} \\ &= \frac{\beta^n}{(\alpha-1)\cdots(\alpha-n)}. \end{aligned}$$

In particular, for $\alpha > 1$

$$E(X) = \frac{\beta}{\alpha-1}$$

and for $\alpha > 2$

$$E(X^2) = \frac{\beta^2}{(\alpha-1)(\alpha-2)}$$

and so for $\alpha > 2$

$$Var(X) = E(X^2) - E(X)^2 = \frac{\beta^2}{(\alpha-1)^2(\alpha-2)}.$$

4 Conjugate prior for exponential likelihood

Finally, suppose than an observation $X \mid \mu \sim \text{exponential}(\mu)$ and *a priori* $\mu \sim \text{IG}(\alpha, \beta)$. By Bayes' theorem, the posterior distribution on μ given an observation $X = x$ is proportional to

$$\frac{1}{\mu}\exp(-x/\mu)\frac{1}{\mu^{\alpha+1}}\exp(-\beta/\mu) = \frac{1}{\mu^{\alpha+2}}\exp(-(\beta+x)/\mu).$$

When normalized to be a probability distribution, the result is an $\text{IG}(\alpha+1, \beta+x)$ distribution. In general, after observing $x_1, x_2, \ldots, x_n$, the posterior distribution on μ is $\text{IG}(\alpha+n, \beta+\sum_{i=1}^n x_i)$.

The motivation for parameterizing the inverse gamma distribution the way we do is to make the posterior distribution have the simple form above.

This document available at `http://www.johndcook.com/inverse_gamma.pdf`.