# Multi-index notation

John D. Cook

September 18, 2008

## 1 Introduction

Multi-index notation makes multi-variable generalizations of familiar one-variable theorems easier to remember. We will give two examples: generalizing the binomial theorem to the multinomial theorem and generalizing Taylor series from one variable to multiple variables.

Multi-index notation is well-known in some areas of mathematics but practically unheard of in others. Unfortunately, it is seldom introduced at an undergraduate level even though undergraduates first learning multi-variable theorems could benefit the most from notation that makes the theorems more memorable.

A **multi-index** $\alpha = (\alpha_1, \ldots, \alpha_n)$ is an $n$-tuple of non-negative integers. The norm of a multi-index is defined to by

$$|\alpha| = \alpha_1 + \cdots + \alpha_n.$$

For a vector $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$, define

$$x^\alpha = \prod_{i=1}^{n} x_i^{\alpha_i}.$$

Also, define the factorial of a multi-index $\alpha$ by

$$\alpha! = \prod_{i=1}^{n} \alpha_i!.$$

For example, if $\alpha = (2, 3, 1)$ then $|\alpha| = 2+3+1 = 6$ and $\alpha! = 2!\,3!\,1! = 12$. If $x = (4, -1, 2)$ then $x^\alpha = 4^2\,(-1)^3\,2^1 = -32$.

## 2 Binomial and multinomial theorems

Using the definitions above, the multinomial theorem becomes

$$(x_1 + x_2 + \cdots + x_n)^m = \sum_{|\alpha|=m} \frac{m!}{\alpha!}(x_1, x_2, \cdots, x_n)^\alpha.$$

Let's see how this reduces to the binomial theorem when $n = 2$. A multi-index of length two is just a pair of non-negative integers. The sum over all multi-indexes of length 2 with norm $m$ is the set of all pairs of the form $(j, m - j)$ where $0 \leq j \leq m$. Thus

$$\begin{aligned}(x_1 + x_2)^m &= \sum_{|\alpha|=m} \frac{m!}{\alpha!}(x_1, x_2)^\alpha \\ &= \sum_{j=0}^{m} \frac{m!}{j!\,(m-j)!} x_1^j x_2^{m-j}\end{aligned}$$

Now let's see how the notation applies to three variables. A multi-index of length three and norm $m$ is a triple of non-negative integers $(i, j, k)$ such that $i + j + k = m$.

$$\begin{aligned}(x_1 + x_2 + x_3)^m &= \sum_{|\alpha|=m} \frac{m!}{\alpha!}(x_1, x_2, x_3)^\alpha \\ &= \sum_{i+j+k=m} \frac{m!}{i!\,j!\,k!} x_1^i x_2^j x_3^k\end{aligned}$$

## 3 Taylor's theorem

Next, define the multi-index power of a derivative operator analogously to the multi-index power of a vector. That is,

$$D^\alpha = \left(\frac{\partial}{\partial x_1}\right)^{\alpha_1} \cdots \left(\frac{\partial}{\partial x_n}\right)^{\alpha_n}.$$

Using this notation, Taylor's theorem in several variables can be written

$$f(x) = \sum_{|\alpha|=0}^{\infty} \frac{D^\alpha f(x_0)(x - x_0)^\alpha}{\alpha!}.$$

Here $x$ and $x_0$ are vectors of length $n$ and the sum is over all multi-indexes of length $n$ and of all norms. The $m$th order Taylor approximation sums only over multi-indexes of norm less than or equal to $m$.

Even in only two variables, multi-index notation greatly cleans up the statement of Taylor's theorem. The statement of the theorem in most calculus books is so typographically unwieldy that these books seldom go beyond the second order terms, leaving the general term of the sum unclear.

To appreciate how much is packed into the succinct notation above, let's unpack some of the terms for a function of two variables. Let $x = (u, v)$ and $x_0 = (u_0, v_0)$. Also, define $(h, k) = (u - u_0, v - v_0)$. There is only one multi-index of length two with norm zero: (0, 0). The zeroth derivative of a function is just that function, so the term corresponding to $|\alpha| = 0$ is simply

$$f(u_0, v_0).$$

There are two multi-indexes corresponding to $|\alpha| = 1$, (1, 0) and (0, 1), and so these contribute

$$\frac{\partial}{\partial u} f(u_0, v_0)(h, k)^{(1,0)} + \frac{\partial}{\partial v} f(u_0, v_0)(h, k)^{(0,1)}$$

which reduces to

$$f_u(u_0, v_0)h + f_v(u_0, v_0)k.$$

The factorial terms have dropped out since $\alpha! = 1$ for (1, 0) and (0, 1).

Next we unpack the quadratic term. There are three multi-indexes corresponding to $|\alpha| = 2$: (2, 0), (1, 1), and (0, 2). These three terms contribute

$$\frac{1}{2} f_{uu}(u_0, v_0)h^2 + f_{uv}(u_0, v_0)hk + \frac{1}{2} f_{vv}(u_0, v_0)k^2$$

to the series. This may look slightly unfamiliar since calculus books more often write the above term as

$$\frac{1}{2}\left(f_{uu}(u_0, v_0)h^2 + 2f_{uv}(u_0, v_0)hk + f_{vv}(u_0, v_0)k^2\right).$$

Finally, we'll unpack the cubic term. There are four multi-indexes of length 2 and norm 3: (3,0), (2, 1), (1, 2) and (0, 3). The cubic term in Taylor's theorem is thus

$$\frac{1}{6} f_{uuu}(u_0, v_0)h^3 + \frac{1}{2} f_{uuv}(u_0, v_0)h^2k + \frac{1}{2} f_{uvv}(u_0, v_0)hk^2 + \frac{1}{6} f_{vvv}(u_0, v_0)k^3.$$

## 4 Conclusion

The only drawback to multi-index notation is that it may make theorem statements too simple; someone seeing the notation before seeing the more verbose alternatives may not appreciate all the activity going on behind the simple symbols.

This document is available at `http://www.johndcook.com`.