Upper and lower bounds for the normal distribution function

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Let Z be a standard normal random variable. These notes present upper and lower bounds for the complementary cumulative distribution function

$$\Phi^c(t) = P(Z > t) = \frac{1}{\sqrt{2\pi}} \int_t^\infty e^{-x^2/2}\, dx.$$

We prove simple bounds first then state improved bounds without proof.

An upper bound is easy to obtain. Since $x/t > 1$ for x in (t, ∞), we have

$$\begin{aligned}
\Phi^c(t) &= \frac{1}{\sqrt{2\pi}} \int_t^\infty e^{-x^2/2}\, dx && (1)\\
&< \frac{1}{\sqrt{2\pi}} \int_t^\infty \frac{x}{t} e^{-x^2/2}\, dx && (2)\\
&= \frac{1}{\sqrt{2\pi}} \frac{1}{t} e^{-t^2/2}. && (3)
\end{aligned}$$

We can also show there is a lower bound

$$\Phi^c(t) > \frac{1}{\sqrt{2\pi}} \frac{t}{t^2+1} e^{-t^2/2}. \qquad (4)$$

To prove inequality (4), define

$$g(t) = \Phi^c(t) - \frac{1}{\sqrt{2\pi}} \frac{t}{t^2+1} e^{-t^2/2}.$$

We will show that $g(t)$ is always positive. Clearly $g(0) > 0$. From the derivative

$$g'(t) = -2\frac{e^{-t^2/2}}{(t^2+1)^2}$$

we see that g is strictly decreasing. Since $\lim_{t\to\infty} g(t) = 0$, g must always be positive.

Combining inequalities (3) and (4),

$$\frac{t}{t^2+1} < \sqrt{2\pi}e^{t^2/2}\Phi^c(t) < \frac{1}{t}. \tag{5}$$

Abramowitz and Stegun (Handbook of Mathematical Functions, Dover) give bounds on the error function from which we can derive different bounds on the normal distribution. Formula 7.1.13 from Abramowitz and Stegun reads

$$\frac{1}{x+\sqrt{x^2+2}} < e^{x^2}\int_x^\infty e^{-t^2}\,dt \leq \frac{1}{x+\sqrt{x^2+4/\pi}}. \tag{6}$$

Let $t = \sqrt{2}x$. Then the inequality (6) implies

$$\frac{1}{t+\sqrt{t^2+4}} < \sqrt{\frac{\pi}{2}}e^{t^2/2}\Phi^c(t) \leq \frac{1}{t+\sqrt{t^2+8/\pi}}. \tag{7}$$

Both the upper and lower bounds in inequality (7) are uniformly better than those in inequality (5).

http://www.johndcook.com/normalbounds.pdf.