The "pqr" theorem

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Suppose p, q and r are seminorms on V. Suppose $p+r$ and $p+q$ are norms. Define

$$\begin{aligned} \|u\|_{pr} &\equiv p(u) + r(u) \\ \|u\|_{pq} &\equiv p(u) + q(u) \\ \|u\|_{r} &\equiv r(u) \end{aligned}$$

and let V_{pr}, V_{pq} and V_r denote V with each of these (semi-) norms. If V_{pr} is reflexive, embeds continuously into V_{pq}, and embeds compactly into V_r, then V_{pq} embeds continuously into V_{pr} and so the pq and pr norms are equivalent.

Proof. Suppose V_{pq} does not imbed into V_{pr}. Then there exists a sequence v_n such that $\|v - n\|_{pr} = 1$ and $\|v - n\|_{pq} \to 0$. Since $\{v_n\}$ is a bounded sequence in a reflexive Banach space, it has a subsequence $v_i \rightharpoonup v$ in V_{pr}. By compactness, $v_i \to v$ in V_r. By continuity, $v_i \rightharpoonup v$ in V_{pq}, but $v_i \to 0$ in V_{pq} and so v must be 0. From the definitions,

$$\|v_i\|_{pq} + \|v_i\|_r \geq \|v_i\|_{pr}.$$

Since $v_i \to 0$ in V_{pq} and V_r, the left side goes to zero and so $\|v_i\| \to 0$. Since every subsequence of v_n has a further subsequence that converges to 0, it must be the case that the original sequence $\{v_n\}$ goes to 0. □

A typical application would be to show that the principal part of the $W^{1,p}$ norm bounds the full norm under certain circumstances. For example, suppose Ω is bounded. If φ has zero average, one can show that the norm of gradient of φ controls the norm of φ by setting

$$\begin{aligned} p(\varphi) &= \|\vec{\nabla}\varphi\|_{L^p(\Omega)}, \\ r(\varphi) &= \|\varphi\|_{L^p(\Omega)}, \\ q(\varphi) &= |\int_\Omega \varphi \, dx|. \end{aligned}$$

(Basically, the principle part of the $W^{1,p}$ norm almost controls the L^p part. The seminorm q need only be strong enough to distinguish constant functions. For example q could be the integral of φ over some subset of Ω or $\partial\Omega$ of positive measure.)