Notes on functions of regular variation

John D. Cook

April 29, 2009

The theory of functions of regular variation was developed by Jovan Karamata starting in 1930. His original application was Taubarian theorems. Functions of regular variation have subsequently been applied to the study of stationary distributions in probability and to asymptotic behavior of differential equations.

Let $U : [0, \infty) \to [0, \infty)$ be a measurable function. We say U has regular variation of order ρ at infinity if there exists a real number ρ such that for every $x > 0$

$$\lim_{t \to \infty} \frac{U(tx)}{U(x)} = t^{\rho}.$$

The definition can be extended to allow $\rho = \pm\infty$ where

$$x^{\pm\infty} = \lim_{t \to \pm\infty} x^{t}.$$

If $\rho = 0$, the function U is said to be *slowly varying at infinity*. If $\rho = \pm\infty$, the function U is said to be *rapidly varying at infinity*.

Examples: The functions x^{ρ} and $x^{\rho} \log(1 + x)$ have regular variation of order ρ at infinity. The function $\arctan x$ is slowly varying at infinity, *i.e.* has order $\rho = 0$. The functions e^{x} and e^{-x} are rapidly varying at infinity with orders $+\infty$ and $-\infty$ respectively.

Here is a condition that appears to be weaker than regular variation but is actually equivalent.

Let $U : [0, \infty) \to [0, \infty)$ be a measurable function. Suppose for every $x > 0$

$$\lim_{t\to\infty} \frac{U(tx)}{U(x)} = h(x)$$

where $0 < h(x) < \infty$. Then $h(x) = x^\rho$ for some $-\infty < \rho < \infty$.

Here is a sufficient condition for regular variation using integer sequences.

A monotone function $U : [0, \infty) \to [0, \infty)$ has regular variation at infinity if there exist two integers, m_1 and m_2 such that $\log(m_1)/\log(m_2)$ is irrational and

$$\lim_{n\to\infty} \frac{U(nm)}{U(n)} = m^\rho$$

for some real ρ for $m = m_1$ and $m = m_2$. Here $n \to \infty$ through integer values. This is Theorem 1.1.2 from *On regular variation and its application to the weak convergence of sample extremes* by L. de Haan.

The following representation theorem comes from Katamata.

Suppose U is regularly varying with exponent ρ and locally integrable. Then

$$U(x) = c(x) \exp\left(\int_1^x \frac{a(t)}{t}\, dt\right)$$

for some functions $a(x) : [0, \infty) \to (-\infty, \infty)$ and $c(x) : [0, \infty) \to [0, \infty)$. We have the limits

$$\lim_{x\to\infty} a(x) = \rho$$

and

$$\lim_{x\to\infty} c(x) = c_0$$

for some $0 < c_0 < \infty$.