Separation of the Helmholtz Equation

Coordinate-by-Coordinate Forms with Polynomial Coefficients

John D. Cook

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Abstract

For each of eleven coordinate systems we write the Helmholtz equation $\nabla^2 u + k^2 u = 0$ explicitly, and then, where the coefficients involve trigonometric, hyperbolic, or elliptic functions, we exhibit a change of variables after which every coefficient is a polynomial. The rationalizing substitution is essentially the same throughout: replace each angular variable by its cosine, sine, or hyperbolic cosine, which converts operators of the type $\frac{1}{\sin\theta}\partial_\theta(\sin\theta\,\partial_\theta)$ into $\partial_s(P(s)\,\partial_s)$ with P quadratic. Two of the eleven systems, bipolar cylindrical and bispherical, do not separate the Helmholtz equation; for these we give the Laplace equation $\nabla^2 u = 0$ as well, which they do separate.

Preliminary remark on the classical list

The Eisenhart–Morse–Feshbach theorem states that $\nabla^2 u + k^2 u = 0$ separates in exactly eleven coordinate systems: Cartesian, circular cylindrical, elliptic cylindrical, parabolic cylindrical, spherical, prolate spheroidal, oblate spheroidal, rotational (circular) parabolic, conical, ellipsoidal, and paraboloidal.

Bipolar cylindrical and bispherical coordinates do *not* belong to this list. They R-separate the Laplace equation $\nabla^2 u = 0$, but the modulating factor R cannot absorb a k^2 term, so separation fails for $k \neq 0$. Sections 2 and 8 therefore record the Helmholtz form for completeness, flag the failure, and then give the Laplace equation, which is the case of genuine interest for those two systems.

Throughout, $u = u(\cdot,\cdot,\cdot)$ and subscripts denote partial derivatives. "Rationalized form" always means: multiplied through by the least common denominator, so that the equation has polynomial coefficients.

Contents

1 Circular cylindrical coordinates (r, θ, z)

Definition. $x = r\cos\theta$, $y = r\sin\theta$, $z = z$; scale factors $h_r = 1$, $h_\theta = r$, $h_z = 1$.

Helmholtz equation.

$$u_{rr} + \frac{1}{r}\,u_r + \frac{1}{r^2}\,u_{\theta\theta} + u_{zz} + k^2 u = 0. \tag{1}$$

Rationalized form. No change of variables is needed; multiplying by r^2 clears the only denominator:

$$r^2 u_{rr} + r\,u_r + u_{\theta\theta} + r^2 u_{zz} + k^2 r^2 u = 0. \tag{2}$$

Separated equations. With $u = R(r)e^{im\theta}e^{\pm i\lambda z}$,

$$r^2 R'' + rR' + \big[(k^2 - \lambda^2)r^2 - m^2\big]R = 0,$$

which is essentially Bessel's equation.

2 Bipolar cylindrical coordinates (σ, τ, z)

Definition.

$$x = \frac{a\sinh\tau}{\cosh\tau - \cos\sigma}, \qquad y = \frac{a\sin\sigma}{\cosh\tau - \cos\sigma}, \qquad z = z,$$

with $h_\sigma = h_\tau = \dfrac{a}{H}$, $h_z = 1$, where $H = \cosh\tau - \cos\sigma$.

Helmholtz equation.

$$u_{\sigma\sigma} + u_{\tau\tau} + \frac{a^2}{(\cosh\tau - \cos\sigma)^2}\big(u_{zz} + k^2 u\big) = 0. \tag{3}$$

Not separable. The variables σ and τ cannot be split unless the entire bracket $u_{zz} + k^2 u$ is annihilated, i.e. unless k^2 is absorbed into the z-separation constant. Genuine separation occurs only for the two-dimensional Laplace case $k = 0$, $\partial_z u = 0$.

Rationalized form. Substitute

$$s = \cos\sigma, \qquad t = \cosh\tau,$$

under which $\partial_\sigma^2 \mapsto (1-s^2)\partial_s^2 - s\,\partial_s$ and $\partial_\tau^2 \mapsto (t^2-1)\partial_t^2 + t\,\partial_t$, and $H = t - s$. Multiplying by $(t-s)^2$:

$$(t-s)^2\Big[(1-s^2)u_{ss} - s\,u_s + (t^2-1)u_{tt} + t\,u_t\Big] + a^2\big(u_{zz} + k^2 u\big) = 0. \tag{4}$$

Laplace equation ($k = 0$).

$$u_{\sigma\sigma} + u_{\tau\tau} + \frac{a^2}{H^2}\,u_{zz} = 0 \qquad \Longrightarrow \qquad (t-s)^2\Big[(1-s^2)u_{ss} - s\,u_s + (t^2-1)u_{tt} + t\,u_t\Big] + a^2 u_{zz} = 0. \tag{5}$$

If in addition $u_{zz} = 0$, the prefactor $(t-s)^2$ divides out entirely, exhibiting (σ, τ) as conformal coordinates:

$$(1-s^2)u_{ss} - s\,u_s + (t^2-1)u_{tt} + t\,u_t = 0, \tag{6}$$

with solutions $e^{\pm n\sigma}$, $e^{\pm n\tau}$, i.e. Chebyshev-type functions of s and t.

3 Elliptic cylindrical coordinates (ξ, η, z)

Definition. $x = a\cosh\xi\cos\eta$, $y = a\sinh\xi\sin\eta$, $z = z$; $h_\xi = h_\eta = a\sqrt{\sinh^2\xi + \sin^2\eta}$, $h_z = 1$.

Helmholtz equation.

$$u_{\xi\xi} + u_{\eta\eta} + a^2\big(\sinh^2\xi + \sin^2\eta\big)\big(u_{zz} + k^2 u\big) = 0. \tag{7}$$

Rationalized form. Substitute

$$t = \cosh\xi, \qquad s = \cos\eta,$$

so that $\sinh^2\xi + \sin^2\eta = t^2 - s^2$:

$$(t^2-1)u_{tt} + t\,u_t + (1-s^2)u_{ss} - s\,u_s + a^2(t^2-s^2)\big(u_{zz} + k^2 u\big) = 0. \tag{8}$$

Separated equations. With $u = T(t)S(s)e^{\pm i\lambda z}$ and $c^2 = a^2(k^2 - \lambda^2)$,

$$(t^2-1)T'' + tT' + \big(\mu + c^2 t^2\big)T = 0,$$

which is Mathieu's equation in *algebraic form*; the s-equation is its companion on $[-1, 1]$.

4 Parabolic cylindrical coordinates (ξ, η, z)

Definition. $x = \xi\eta$, $y = \frac{1}{2}(\xi^2 - \eta^2)$, $z = z$; $h_\xi = h_\eta = \sqrt{\xi^2 + \eta^2}$, $h_z = 1$.

Helmholtz equation.

$$u_{\xi\xi} + u_{\eta\eta} + (\xi^2 + \eta^2)\big(u_{zz} + k^2 u\big) = 0. \tag{9}$$

Rationalized form. Already polynomial; no substitution required.

Separated equations. Each factor satisfies a Weber (parabolic cylinder) equation

$$X'' + \big(\pm\beta + \gamma\,\xi^2\big)X = 0, \qquad \gamma = k^2 - \lambda^2.$$

5 Circular (rotational) parabolic coordinates (ξ, η, φ)

Definition. $x = \xi\eta\cos\varphi$, $y = \xi\eta\sin\varphi$, $z = \frac{1}{2}(\xi^2 - \eta^2)$; $h_\xi = h_\eta = \sqrt{\xi^2 + \eta^2}$, $h_\varphi = \xi\eta$.

Helmholtz equation.

$$u_{\xi\xi} + \frac{1}{\xi}u_\xi + u_{\eta\eta} + \frac{1}{\eta}u_\eta + \frac{\xi^2 + \eta^2}{\xi^2\eta^2}\,u_{\varphi\varphi} + k^2(\xi^2 + \eta^2)\,u = 0. \tag{10}$$

Rationalized form. Multiplying by $\xi^2\eta^2$:

$$\eta^2\big(\xi^2 u_{\xi\xi} + \xi u_\xi\big) + \xi^2\big(\eta^2 u_{\eta\eta} + \eta u_\eta\big) + (\xi^2 + \eta^2)u_{\varphi\varphi} + k^2\xi^2\eta^2(\xi^2 + \eta^2)u = 0. \tag{11}$$

No change of variables is required.

Separated equations. With $u = X(\xi)Y(\eta)e^{im\varphi}$,

$$\xi^2 X'' + \xi X' + \big(k^2\xi^4 + \beta\xi^2 - m^2\big)X = 0,$$

and the same with $\beta \to -\beta$ for Y.

6 Confocal paraboloidal coordinates (λ, μ, ν)

Definition. λ, μ, ν are the roots of

$$\frac{x^2}{a - \theta} + \frac{y^2}{b - \theta} + \theta = 2z, \qquad \lambda < a < \mu < b < \nu,$$

so that

$$x^2 = \frac{(a - \lambda)(a - \mu)(a - \nu)}{b - a}, \qquad y^2 = \frac{(b - \lambda)(b - \mu)(b - \nu)}{a - b}, \qquad z = \tfrac{1}{2}(\lambda + \mu + \nu - a - b).$$

Helmholtz equation. Writing $R(\theta) = (a - \theta)(b - \theta)$ and letting θ range over $\{\lambda, \mu, \nu\}$ with θ', θ'' the other two,

$$\sum_{\theta} \frac{4}{(\theta - \theta')(\theta - \theta'')}\Big[R(\theta)u_{\theta\theta} + \tfrac{1}{2}R'(\theta)u_\theta\Big] + k^2 u = 0. \tag{12}$$

Rationalized form. Clearing denominators with $\Delta = (\lambda - \mu)(\mu - \nu)(\nu - \lambda)$:

$$\begin{aligned} 4(\nu - \mu)\Big[R(\lambda)u_{\lambda\lambda} + \tfrac{1}{2}R'(\lambda)u_\lambda\Big] + 4(\lambda - \nu)\Big[R(\mu)u_{\mu\mu} + \tfrac{1}{2}R'(\mu)u_\mu\Big] \\ + 4(\mu - \lambda)\Big[R(\nu)u_{\nu\nu} + \tfrac{1}{2}R'(\nu)u_\nu\Big] + k^2\Delta\,u = 0. \end{aligned} \tag{13}$$

Already polynomial; no change of variables is required.

Separated equations. Each of the three factors satisfies the same *Baer wave equation*

$$4\Big[R(\theta)X'' + \tfrac{1}{2}R'(\theta)X'\Big] + \big(k^2\theta^2 + c_1\theta + c_2\big)X = 0.$$

For $k = 0$ this reduces to Baer's equation, a Fuchsian equation with regular singular points at $\theta = a,\, b,\, \infty$.

7 Spherical coordinates (r, θ, φ)

Definition. $x = r\sin\theta\cos\varphi$, $y = r\sin\theta\sin\varphi$, $z = r\cos\theta$; $h_r = 1$, $h_\theta = r$, $h_\varphi = r\sin\theta$.

Helmholtz equation.

$$u_{rr} + \frac{2}{r}u_r + \frac{1}{r^2}\left[u_{\theta\theta} + \cot\theta\, u_\theta + \frac{1}{\sin^2\theta}u_{\varphi\varphi}\right] + k^2 u = 0. \tag{14}$$

Rationalized form. Substitute $s = \cos\theta$, so that $\frac{1}{\sin\theta}\partial_\theta(\sin\theta\,\partial_\theta) \mapsto \partial_s\big((1-s^2)\partial_s\big)$. Multiplying by $r^2(1-s^2)$:

$$(1-s^2)\big(r^2 u_{rr} + 2r\,u_r + k^2 r^2 u\big) + (1-s^2)^2 u_{ss} - 2s(1-s^2)u_s + u_{\varphi\varphi} = 0. \tag{15}$$

Separated equations. Spherical Bessel functions in r; associated Legendre functions $P_n^m(s)$, $Q_n^m(s)$ in s.

8 Bispherical coordinates (σ, τ, φ)

Definition.

$$x = \frac{a\sin\sigma\cos\varphi}{H}, \qquad y = \frac{a\sin\sigma\sin\varphi}{H}, \qquad z = \frac{a\sinh\tau}{H}, \qquad H = \cosh\tau - \cos\sigma,$$

with $h_\sigma = h_\tau = a/H$ and $h_\varphi = a\sin\sigma/H$.

Helmholtz equation.

$$\frac{H^3}{a^2\sin\sigma}\left[\partial_\sigma\left(\frac{\sin\sigma}{H}u_\sigma\right) + \partial_\tau\left(\frac{\sin\sigma}{H}u_\tau\right) + \frac{u_{\varphi\varphi}}{H\sin\sigma}\right] + k^2 u = 0. \tag{16}$$

Not separable. With the modulating factor $u = \sqrt{H}\,S(\sigma)T(\tau)\Phi(\varphi)$ the equation R-separates *only* when $k = 0$: the factor $\sqrt{H}$ can absorb the first-order cross terms but not a term proportional to $k^2 u$, whose coefficient scales as H^{-3} rather than H^{-2}.

Rationalized form. Substitute $s = \cos\sigma$, $t = \cosh\tau$ (so $H = t - s$) and multiply by $(t-s)^3(1-s^2)$:

$$\begin{aligned}(1-s^2)(t-s)^2\Big[(1-s^2)u_{ss} - 2s\,u_s + (t^2-1)u_{tt} + t\,u_t\Big]& \\ + (1-s^2)(t-s)\Big[(1-s^2)u_s - (t^2-1)u_t\Big] + (t-s)^2 u_{\varphi\varphi} + a^2k^2(1-s^2)\,u = 0.&\end{aligned} \tag{17}$$

Laplace equation ($k = 0$).

$$\partial_\sigma\left(\frac{\sin\sigma}{H}u_\sigma\right) + \partial_\tau\left(\frac{\sin\sigma}{H}u_\tau\right) + \frac{u_{\varphi\varphi}}{H\sin\sigma} = 0, \tag{18}$$

which in the rational variables $s = \cos\sigma$, $t = \cosh\tau$, after multiplication by $(t-s)^2(1-s^2)$, becomes

$$(1-s^2)(t-s)\Big[(1-s^2)u_{ss} - 2s\,u_s + (t^2-1)u_{tt} + t\,u_t\Big] \\ + (1-s^2)\Big[(1-s^2)u_s - (t^2-1)u_t\Big] + (t-s)\,u_{\varphi\varphi} = 0. \tag{19}$$

This is the Helmholtz form with one factor of $(t-s)$ removed, which is exactly what makes R-separation possible.

Separated equations. With $u = \sqrt{\cosh\tau - \cos\sigma}\; S(\sigma)T(\tau)e^{im\varphi}$,

$$\big[(t^2-1)T'\big]' = \big(n+\tfrac{1}{2}\big)^2 T, \qquad \text{i.e.} \qquad T = e^{\pm(n+1/2)\tau},$$

and S satisfies the associated Legendre equation in s.

9 Oblate spheroidal coordinates (ξ, η, φ)

Definition.

$$x = a\cosh\xi\cos\eta\cos\varphi, \qquad y = a\cosh\xi\cos\eta\sin\varphi, \qquad z = a\sinh\xi\sin\eta,$$

with $h_\xi = h_\eta = a\sqrt{\sinh^2\xi + \sin^2\eta}$ and $h_\varphi = a\cosh\xi\cos\eta$.

Helmholtz equation.

$$\frac{1}{\sinh^2\xi + \sin^2\eta}\left[\frac{1}{\cosh\xi}\partial_\xi\big(\cosh\xi\,u_\xi\big) + \frac{1}{\cos\eta}\partial_\eta\big(\cos\eta\,u_\eta\big)\right] + \frac{u_{\varphi\varphi}}{\cosh^2\xi\,\cos^2\eta} + k^2a^2u = 0. \tag{20}$$

Rationalized form. Substitute

$$\zeta = \sinh\xi, \qquad s = \sin\eta,$$

under which the two bracketed operators become $\partial_\zeta\big((1+\zeta^2)\partial_\zeta\big)$ and $\partial_s\big((1-s^2)\partial_s\big)$, while $\sinh^2\xi + \sin^2\eta = \zeta^2 + s^2$. Multiplying by $(\zeta^2+s^2)(1+\zeta^2)(1-s^2)$:

$$(1+\zeta^2)(1-s^2)\Big[(1+\zeta^2)u_{\zeta\zeta} + 2\zeta\,u_\zeta + (1-s^2)u_{ss} - 2s\,u_s\Big] \\ + (\zeta^2+s^2)\,u_{\varphi\varphi} + k^2a^2(\zeta^2+s^2)(1+\zeta^2)(1-s^2)\,u = 0. \tag{21}$$

Separated equations. The oblate spheroidal wave equation, obtained from the prolate one by $\zeta \to i\zeta$, $c \to -ic$.

10 Prolate spheroidal coordinates (ξ, η, φ)

Definition.

$$x = a\sinh\xi\sin\eta\cos\varphi, \qquad y = a\sinh\xi\sin\eta\sin\varphi, \qquad z = a\cosh\xi\cos\eta,$$

with $h_\xi = h_\eta = a\sqrt{\sinh^2\xi + \sin^2\eta}$ and $h_\varphi = a\sinh\xi\sin\eta$.

Helmholtz equation.

$$\frac{1}{\sinh^2\xi+\sin^2\eta}\left[\frac{1}{\sinh\xi}\partial_\xi\big(\sinh\xi\,u_\xi\big)+\frac{1}{\sin\eta}\partial_\eta\big(\sin\eta\,u_\eta\big)\right]+\frac{u_{\varphi\varphi}}{\sinh^2\xi\,\sin^2\eta}+k^2a^2u=0. \tag{22}$$

Rationalized form. Substitute

$$\zeta=\cosh\xi,\qquad s=\cos\eta,$$

so that $\sinh^2\xi+\sin^2\eta=\zeta^2-s^2$, and multiply by $(\zeta^2-s^2)(\zeta^2-1)(1-s^2)$:

$$\begin{aligned}(\zeta^2-1)(1-s^2)\Big[(\zeta^2-1)u_{\zeta\zeta}+2\zeta\,u_\zeta+(1-s^2)u_{ss}-2s\,u_s\Big]&\\ +(\zeta^2-s^2)\,u_{\varphi\varphi}+k^2a^2(\zeta^2-s^2)(\zeta^2-1)(1-s^2)\,u=0.&\end{aligned} \tag{23}$$

Separated equations. The *spheroidal wave equation*, with $c=ka$:

$$\frac{d}{d\zeta}\Big[(\zeta^2-1)Z'\Big]+\left(-\Lambda+c^2\zeta^2-\frac{m^2}{\zeta^2-1}\right)Z=0.$$

For $k=0$ the $c^2\zeta^2$ term vanishes and this is Legendre's associated equation.

11 Conical (sphero-conal) coordinates (r,μ,ν)

Definition. $x^2+y^2+z^2=r^2$ together with

$$\frac{x^2}{\theta^2}+\frac{y^2}{\theta^2-b^2}+\frac{z^2}{\theta^2-c^2}=0,\qquad \theta=\mu,\nu,\qquad \nu^2<c^2<\mu^2<b^2,$$

with

$$h_r=1,\qquad h_\mu^2=\frac{r^2(\mu^2-\nu^2)}{(b^2-\mu^2)(\mu^2-c^2)},\qquad h_\nu^2=\frac{r^2(\mu^2-\nu^2)}{(b^2-\nu^2)(c^2-\nu^2)}.$$

Helmholtz equation. Writing $\Theta(t)=(b^2-t^2)(t^2-c^2)$,

$$u_{rr}+\frac{2}{r}u_r+\frac{1}{r^2(\mu^2-\nu^2)}\Big[\Theta(\mu)u_{\mu\mu}+\tfrac12\Theta'(\mu)u_\mu-\Theta(\nu)u_{\nu\nu}-\tfrac12\Theta'(\nu)u_\nu\Big]+k^2u=0. \tag{24}$$

Rationalized form.

$$(\mu^2-\nu^2)\big(r^2u_{rr}+2r\,u_r+k^2r^2u\big)+\Theta(\mu)u_{\mu\mu}+\tfrac12\Theta'(\mu)u_\mu-\Theta(\nu)u_{\nu\nu}-\tfrac12\Theta'(\nu)u_\nu=0. \tag{25}$$

This *is* the rational form. Conical coordinates are frequently parametrized by Jacobi elliptic functions, $\mu=c\,\mathrm{ns}(\alpha,\kappa)$, $\nu=c\,\mathrm{sn}(\beta,\kappa)$, in which case the angular equations appear as Lamé's equation in Jacobian form with sn^2 coefficients; passing back to the algebraic variables μ,ν above is precisely the change of variables that rationalizes them.

Separated equations. Radial: spherical Bessel (or r^n, r^{-n-1} when $k=0$). Angular: both factors satisfy *Lamé's equation in algebraic form*

$$\Theta(t)M''+\tfrac12\Theta'(t)M'+\big(h-n(n+1)t^2\big)M=0,$$

whose admissible solutions are the Lamé polynomials (ellipsoidal surface harmonics).

Summary of substitutions

System	Substitution	Needed?	Helmholtz separable?
Circular cylindrical	—	no	yes
Bipolar cylindrical	$s = \cos\sigma,\ t = \cosh\tau$	yes	no ($k = 0$ only, 2-D)
Elliptic cylindrical	$t = \cosh\xi,\ s = \cos\eta$	yes	yes
Parabolic cylindrical	—	no	yes
Circular parabolic	—	no	yes
Confocal paraboloidal	—	no	yes
Spherical	$s = \cos\theta$	yes	yes
Bispherical	$s = \cos\sigma,\ t = \cosh\tau$	yes	no (R-sep. for $k = 0$)
Oblate spheroidal	$\zeta = \sinh\xi,\ s = \sin\eta$	yes	yes
Prolate spheroidal	$\zeta = \cosh\xi,\ s = \cos\eta$	yes	yes
Conical	algebraic μ, ν (not Jacobian)	if elliptic	yes

Every angular variable enters through an operator of the type $\frac{1}{\sin}\partial(\sin\,\partial)$, and setting the new variable equal to the cosine (or sine, or hyperbolic cosine) of the old one converts these into $\partial(P\,\partial)$ with P quadratic. The azimuthal variable φ and the translational variable z never require transformation, since they appear only as $u_{\varphi\varphi}$ and u_{zz} with constant coefficients.