Student-t as a mixture of normals

Claim: Let $X \mid W$ be normal with mean 0 and variance W. Let $W \sim$ inverse gamma$(\nu/2, \nu/2)$. Then the marginal distribution on X is Student-t with ν degrees of freedom.

Proof

$$\begin{aligned}
f_X(x) &= \int_0^\infty f_{X \mid W}(x)\, f_W(w)\, dw \\
&\propto \int_0^\infty \frac{1}{\sqrt{w}} \exp\left(-\frac{x^2}{2w}\right) w^{-\frac{\nu}{2}-1} \exp\left(-\frac{\nu}{2w}\right)\, dw \\
&= \int_0^\infty w^{-\frac{\nu+1}{2}-1} \exp\left(-\frac{x^2+\nu}{2w}\right)\, dw \\
&= \Gamma\left(\frac{\nu+1}{2}\right)\left(\frac{x^2+\nu}{2}\right)^{-\frac{\nu+1}{2}}
\end{aligned}$$

The last line is proportional to the distribution for a Student-t distribution with ν degrees of freedom and so $X \sim t_\nu$. ◊

Some properties of the t distribution are easier to prove in this form than to prove directly.

First, it is easy to show that the t distribution converges to the normal. The expected value of W is $\nu/(\nu-2)$ for all $\nu > 2$. As $\nu \to \infty$, the mean of W goes to 1 and the variance goes to 0. So as ν increases, W becomes concentrated at 1 and X converges to a standard normal random variable.

Also, the variance of the t distribution falls out of the general identity

$$\mathrm{Var}(X) = \mathrm{E}(\mathrm{Var}(X \mid Y)) + \mathrm{Var}(\mathrm{E}(X \mid Y)).$$

Setting $Y = W$, we have

$$\mathrm{Var}(X) = \mathrm{E}(W) + 0 = \frac{\nu}{\nu-2}$$

for $\nu > 2$.