Define $T(n,1) = n$ and for $k > 0$ define

$$T(n,k) = \sum_{i=1}^{n} T(i, k-1).$$

The numbers $T(n,2)$ are the triangle numbers, $T(n,3)$ are the tetrahedral numbers, and $T(n,k)$ are the k-dimensional analogs of the tetrahedral numbers.

There is a simple formula for $T(n,k)$ and the formula can be proved easily using induction. However, the formula appears unmotivated. We can derive a formula for $T(n,k)$ in a more systematic way using the calculus of finite differences.

First, we define the finite difference operator Δ, a discrete analog of the derivative.

$$\Delta f(x) = f(x+1) - f(x).$$

There is a finite difference theorem for sums analogous to the fundamental theorem of calculus for integrals. If $\Delta F(x) = f(x)$ then

$$\sum_{a \leq x < b} f(x) = F(b) - F(a).$$

Next, we define falling and rising powers, also known as factorial powers.

The kth falling power of x is defined as

$$x^{\underline{k}} = x(x-1)(x-2)\cdots(x-k+1).$$

The kth rising power of x is defined as

$$x^{\overline{k}} = x(x+1)(x+2)\cdots(x+k-1).$$

The finite difference operator Δ operates on rising and falling powers analogously to the way the derivative operates on powers:

$$\begin{aligned} \Delta x^{\underline{k}} &= k x^{\underline{k-1}} \\ \Delta x^{\overline{k}} &= k(x+1)^{\overline{k-1}}. \end{aligned}$$

This means we can find the analog of anti-derivatives for falling and rising powers.

$$
\begin{aligned}
\Delta\left(\frac{x^{\underline{k+1}}}{k+1}\right) &= x^{\underline{k}} \\
\Delta\left(\frac{(x-1)^{\overline{k+1}}}{k+1}\right) &= x^{\overline{k}}.
\end{aligned}
$$

Now we are ready to derive the formula for $T(n,k)$. It's well known that the triangular numbers $T(n,2)$ are given by $n(n+1)/2$. The $n(n+1)$ term is a rising power, and so we might suspect that there is a formula for $T(n,k)$ in terms of rising powers. We have $T(n,2) = n^{\overline{2}}/2!$. It is also trivial to check that $T(n,1) = n^{\overline{1}}/1!$. We suspect $T(n,k) = n^{\overline{k}}/k!$.

Note that if we define

$$F(x) = \frac{(x-1)^{\overline{k}}}{k!}$$

then

$$\Delta F(x) = \frac{x^{\overline{k-1}}}{(k-1)!}.$$

We can now prove our formula in a simple calculation.

$$T(n,k) = \sum_{1 \le x < n+1} \frac{x^{\overline{k-1}}}{(k-1)!} = F(n+1) - F(1) = \frac{n^{\overline{k}}}{k!}.$$